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A Population of Partial Differential Equations for Evaluating Methods

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A POPULATION OF PARTIAL DIFFERENTIAL EQUATIONS FOR EVALUATING METHODS

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ABSTRACT

A population of 41 linear two-dimensional elliptic partial differential equations for evaluating numerical methods is presented. Parameters are included in 24 of the problems, allowing a much larger population to be generated. A subset of 20 problems have been included to exhibit various mathematical behaviors of interest. The remaining problems are adapted from "real world" problems in various ways. An attempt is made to structure the problems and measure the complexity of problem, operator, boundary conditions and solution. The set of problems is given in Appendix 1 and a set with precise ELLPACK plus FORTRAN definitions for specific numerical values of the parameters is presented in Appendix 2.

1. PRESENTATION OF THE PROBLEMS.

The problems considering for the evaluation are presented in a uniform format as follows

Format	Example
Problem number	PROBLEM : 5
Source	SOURCE: Artificial
References	REF: [HOUSTIS 78], [HOUSTIS 77]
Operator characteristics	OPER: Constant coef., f entire
Boundary conditions type	B.D. : Dirichlet, Homogeneous
Solution characteristics	SOL: Entire
Parameter description	PARAM: α makes operator more singular without affecting solution
Operator	P.D.E : $4u_{xx} + u_{yy} - \alpha u = f$
Domain	DOM: unit square
Boundary conditions	B.D : $u = 0$
True solution	TRUE: $2(x^2-x)\ (\cos(2\pi y-1))$
Parameter Range	PARAM: $0 \leq \alpha < 100$

Note that "true" is the true solution and any unspecified functions (such as f above) are determined so that true is in fact the solution. Any greek letter (such as α above) is a numerical parameter of the problem and its range is specified.

2. CLASSIFICATION OF THE PROBLEMS

In order to bring some structure into the population of test problems considered we classify them according to source, and features of the problem, operator, boundary conditions and solution.

2.1 Problems Classified by Source.

A. Problems used in previous studies: There are nine problems which have been used in previous studies and which we include so as to provide some comparison with those studies. In some cases we have parametrized the problem and in all cases we have normalized the solution to have a maximum value close to 1.0. The rectangular domains have all been standardized to the unit square $0 \leq x, y \leq 1$.

B. Artificial problems The 20 problems: 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 30, 38, 39, 40, 41 exhibit various mathematical behaviors of interest, but they are not derived from "real world" problems.

C. Problems adapted from the "real world". A persistent difficulty is the desire to have problems which represent the "real world" and the necessity to know their true solutions. We have sometimes defined problems by taking explicit functions as the physical solutions of various problems and then determined the appropriate forcing functions and boundary conditions needed to make the functions the true solutions. Also, series or polynomial solutions generated for various "real world" problems were considered as true solutions by appropriate modification of the real problems. Problems 6, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35, 36, 37, were adapted in this way.

For some problems we take as true solution a finite-Fourier series interpolant of a discrete approximate solution of the "real problem". The operator and boundary conditions are then modified to make this the exact solution.

Many, if not most, physical phenomena are governed by nonlinear operators and we can extend this approach by chosing not only forcing terms but also operator coefficients so as to obtain linear problems whose nature is similar to the "real world" problems. Problems 19, 20, 21, 22, 23, 28, 37 been generated in this manner. The problem source is represented by a numerical parameter P1 = from 0 (artificial problem) to 10 (actual real world problem).

PROBLEM COMPLEXITY CLASSIFICATION: The problem complexity is represented by the numerical parameter P2 whose numerical values are determined according to the following scale.

00 = simplest

30 = some complexity, typical of idealized physical models

50 = rather messy with one or more substantial complications

90 = very complex problem, probably requires special analysis and computational techniques

2.2 Problem Features. We classify the operator, boundary conditions and solution according to smoothness and local variation. A one-dimensional scale 0-100 is used even though there are various rather independent properties that can be called smoothness or local variations. Thus the numerical values of these features are somewhat subjective but we think they are useful for classifying problems.

Smoothness refers to mathematical properties of the functions for operators involved and the scale is, roughly:

00 = entire functions or constants

10 = analytic; very well behaved

30 = very smooth, some higher derivative (5 or so) discontinuity possible

50 = still smooth, third derivative discontinuity possible

70 = not rough to the eye, but possibly only 1 continuous derivative

80 = continuous, functions might be theoretically smooth but rough on a gross scale

90 = possibly discontinuous, nearly singular functions or operators

100 = strong singularities like $1/x$ or $1/x^2$

Local variation refers to how much a function changes (relative to its size) in a small part of its domain. These variations might be oscillations, wave fronts, peaks or boundary layers. The scale we adopt is:

00 = very smooth, uniform

10 = mild variation, probably convex, some non-uniformity
$\sin(2x)$, e^{3x} on $[0,1]$

25 = modest variation or oscillation; mild wave front or peak
$\sin(6x)$, $1/(1+100x^4)$ on $[0,1]$

40 = considerable peak or oscillation; change of magnitude occurs within 10-15% of domain

60 = sharp peaks, wave fronts, boundary layers or oscillations; 100% change in magnitude occurs within 5% of domain

75 = practically a discontinuity in magnitude; continuity observable only with a fine scale examination

90 = actual discontinuity in magnitude; extreme oscillation, step functions or SIN(300x) on [0,1]

The characteristics of a particular problem are represented in the form

* P1.P2, O1.O2, B1.B2, S1.S2

where P1 = Problem type P2 = Problem Complexity

O1 = Operator Smoothness O2 = Operator local variation

B1 = Boundary conditions smoothness B2 = Boundary condition variation

S1 = Solution smoothness S2 = Solution local variation

Preliminary values of the characteristics are given for some of the problems in the equation file EQNFIL of Appendix 2. These values are subject to change as we make the numerical classification more consistent.

2.3 Tables of Problem Characteristics

We also present a simpler classification of the problems in Tables 1-4. Table 1 presents a classification according to operator type and boundary condition type. Table 2 presents a list of characteristics like "peak", "singular", "oscillation", etc. for the 41 problems. Table 3 lists all problems according to properties of the operator and right side of the PDE. Finally, Table 4 lists all problems according to the smoothness of the solution. This smoothness depends on a parameter for some problems and we choose a "normal" value of the parameter for the classification in this table.

TABLE 1 CLASSIFICATION OF PROBLEMS ACCORDING TO OPERATOR AND BOUNDARY CONDITIONS TYPE

OPERATOR:	CONSTANT COEFFICIENTS			NON-CONSTANT COEFFICIENTS		
	DIRICHLET	NEUMANN	MIXED	DIRICHLET	NEUMANN	MIXED
Poisson	3,4,8,9,10, 11,17,33,34, 35,38		31			19
Helryholtz type	5,7,41					
Self-adjoint				1,13,22,23, 30,32		
General	16			36,37,39 6,12,14,15, 18,20,21,24, 26,27,28,29,40		2,25

TABLE 2 LIST OF CHARACTERISTICS OF EACH PROBLEM

(#P = parametrized problem; E = Entire; C = Constant; S = Singular
P = Peak; CC = Comput. Complex; DD = Discont. Deriv.; O = Oscillat ;
D = Discont.; BL = Boundary Layer; WF = Wave Front; SD = Singular Deriv.;
U = Unknown; VS = Variable Smoothness)

PROBLEM NO.	OPERATOR	RIGHT - SIDE	BOUNDARY CONDITIONS D=Dirichlet,M=Mixed,H=Homog	SOLUTION
1	E	E	D-H	E
2	E	E	M-H	E
3	C	E	D-H	E
4P	C	S($\alpha\leq 3$), SD($\alpha>3$)	D-H	SD
5P	C	E	D-H	E
6	E-O	CC-E-O	D-H	O
7	C-S	E	D	BL-S
8	C	P	D	WF-DD
9P	C	P	D-H	P
10	C	C	D	SD
11P	C	O	D	O
12P	O	O	D	O
13	D	DD	D	D
14P	E	E	M	E
15P	S	S	D	BL
16P	C	S	M	SD
17P	C	CC	D	BL
18P	E	CC	D	BL
19P	E-CC	CC	D	E
20P	E	P	M	P
21P	E	E	D	U
22	DD	D	D	E
23P	DD	DD	D	WF
24P	S	S	D	U
25P	O	O	M	U
26P	E	E	D-H	U
27	S	C	D-H	U
28P	CC	CC	D	U
29P	S	C	D	U
30P	VS	VS	D-H	E
31P	C	C	M	E
32P	S	S	D-H	E
33	C	E	D	E
34	C	E	D	O
35	C	C	D	E
36	S	S	D	E
37	E	E	D	E
38P	C	S($\alpha\leq 3$)	D	SD
39P	CC	E-S	D	E-S

TABLE 3 CLASSIFICATION OF PROBLEMS WITH RESPECT TO SMOOTHNESS OF THE OPERATOR

(C = Constants; E = Entire; CC = Computational Complicated;
P = Peak; DD = Discontinuous Derivatives; O = Oscillatory;
S = Singular)

SMOOTHNESS		PROBLEM NUMBER
Operator	Right-Side	
C	C	0,35,31
C	E	,5,7,34,33
E	E	1,2,14,17,18,21,26,37
E	CC	17,18
CC	CC	19,28,39
E	P	28,8,9
DD	DD	22,23,13,30
E	O	11
O	O	12,6,25
E	S	4,16,38
S	E	27,29
S	S	15,24,32,36

TABLE 4 CLASSIFICATION OF PROBLEMS ACCORDING TO SMOOTHNESS OF THE SOLUTION

(SD = Singular Derivatives; BL = Boundary Layer; WF = Wave Front)

Solution Smoothness	Problems Number
E	1,2,3,5,14,19,22,31,32,33,34,35,30, 36,37,39
P	9,20
O	6,11,12
DD	8
SD	4,38,10,13,16
BL	7,15,17,18
WF	23,8

3. PROBLEM FILES

Two files EQNFIL and MACFIL give actual ELLPACK plus FORTRAN implementation of each test problems. The non-parametrized problem of EQNFIL are written as follows:

RECORD Number
Characteristics of the problem (P1.P2, O1,O2, B1,B2, S1,S2)
Problem compatibility (14 items currently checked)
Keywords concerning the operator
P.D.E operator in ELLPACK input format
Keywords concerning the boundary conditions
B.C. in ELLPACK input format
FORTRAN functions for true solution, coefficients of operator and boundary conditions

For the parametrized problems a MACRO is created on with the parameters separately identified by the character "&" following by the name of the parameter.

From each MACRO a record is created in the following way:

RECORD Number
problem characteristics
EXPAND MACRO number /$\eta_1,\eta_2, \ldots,\eta_p$/
where $\eta_1,\ldots,\eta_p$ are numerical values for the P parameters of the problem

The MACRO expansion record is in EQNFIL and the parameterized problems are in MACFIL.

Copies of these files are given in Appendix 2. In Table 5 we graph the record complexity in the scale defined in section 3.

APPENDIX 1

A POPULATION OF PDE's

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PROBLEM: 1

SOURCE: Artificial

REF : [EISEN 73], [HOUSTIS 75] , [HOUSTIS 78]

OPER : Self-adjoint, Non-constant coef., Entire functions

B.C. : Dirichlet, Homogeneous

SOL. : Entire function

PARAM : None

PDE : $(e^{xy}u_x)_x + (e^{-xy}u_y)_y - u/(1+x+y) = f$

DOM : unit square

B.C. : $u = 0$

TRUE : $.75\ e^{xy}\sin(\pi x)\ \sin(\pi y)$

PARAM : None

PROBLEM: 2

SOURCE: Artificial

REF : [HOUSTIS 75], [HOUSTIS 78]

OPER : Non-constant coef., Entire functions

B.C. : Mixed, Homogeneous

SOL. : Entire

PARAM : None

PDE : $u_{xx} + (1+y^2)u_{yy} - u_y - (1+y^2)u_x = f$

DOM : unit square

B.C. : $u - u_N = 0$

TRUE : $.135\ (e^{x+y} + (x^2-x)^2\ \log(1+y^2))$

PARAM : None

PROBLEM: 3

SOURCE: Artificial

REF : [SHULTZ 73] ,[HOUSTIS 75], [HOUSTIS 78]

OPER : Poisson, Entire function

B.C. : Dirichlet, Homogeneous

SOL. : Entire

PARAM :

PDE : $u_{xx} + u_{yy} = 2Cxye^{x}e^{y}(x-x^2)(y-y^2)$

DOM : unit square

B.C : $u = 0$

TRUE : $c\ e^{x}e^{y}\ (x-x^2)(y-y^2)$, $c = 3.$

PARAM :

PROBLEM: 4

SOURCE: Artificial

REF : [HOUSTIS 78]

OPER : Poisson, Right-side function of $x^{\alpha/2-2}$ and $y^{\alpha/2-2}$, Singular for $\alpha \leq 3$

B.C. : Dirichlet, Homogeneous

SOL. : Function of $x^{\alpha/2}$, $y^{\alpha/2}$

PARAM : α adjusts singularity strength

PDE : $u_{xx} + u_{yy} = f$

DOM : unit square

B.C. : $u = 0$

TRUE : $c(x^{\alpha/2}-x)\ (y^{\alpha/2}-y)$, $c = (\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{\alpha}{1-\alpha}})^{-2}$

PARAM : $1.0 < \alpha < 5.0$

PROBLEM: 13

SOURCE: Artificial

OPER : Self-adjoint, non-constant coef., discontinuous coef.

B.C. : Dirichlet

SOL. : Singular Solution derivative

PARAM : None

PDE : $((1 + (x-.4)_+^0)u_x)x + u_{yy} = f$

DOM : unit square

B.C. : $u = g$

TRUE : $\min(x+.3,\ .7+\frac{1}{2}(x-.4))(1 + (y-1)^2 e^{-y})$

PARAM : None

PROBLEM: 14

SOURCE: Artificial

OPER : Non-constant polynomial coef., f entire

B.C. : Mixed

SOL. : Analytic

PARAM : α, β adjust contribution of terms to mixed boundary condition

PDE : $u_{xx} + (1+x^2)\ u_{yy} - y\ u_x = f$

DOM : unit square

BC : $\alpha u + \beta u_x = 0$ for $x=0,1$; $u = g$ for $y=0,1$

TRUE : $\exp(2(x+y)/(2+x-y)-2) + \log_{10}((x+1)/(y+1))$

PARAM : $0 \leq \alpha,\ \beta \leq 1$

PROBLEM: 15

SOURCE: Artificial

OPER : Non-constant coefficients, Singular operator

B.C. : Dirichlet

SOL. : Boundary layer

PARAM : α, ε determine near singularity of operator, β introduces unrelated singularity into solution.

PDE : $u_{xx} + u_{yy} + \alpha/(y+\epsilon)u_y = f$

DOM : unit square

BC : $u = g$

TRUE : $(y^{\beta} + \cos(xy^2)-1)x^2(x-1)^2$

PARAM : $0 \leq \alpha \leq 100,\ 0 \leq \epsilon \leq 1,\ 1.5 \leq \beta \leq 4$

PROBLEM: 16

SOURCE: Artificial

OPER : Constant coefficients

B.C. : Singular boundary conditions

SOL : Singular Solution Derivatives

PARAM : α adjusts the strength of the derivative break in boundary condition

PDE : $u_{xx} + 2\,u_{yy} + 3\,u_x - 4\,u_y - u = f$

DOM : unit square

BC : $u=0$, $y=0$; $u=y$, $x=0$; $u=g$, $x=1$; $u=1.-.8\alpha+\alpha|x-.8|$, $y=1$

TRUE : $y(1-.8\,\alpha^{2-y+\alpha|x-.8|^{2-y}}) + xye^{-xy}(y-1)$

PARAM : $0 \leq \alpha \leq 5$

PROBLEM: 17

SOURCE: Artificial

OPER : Poisson

B.C. : Dirichlet

SOL : Boundary layer

PARV : α, β adjust the shape and strength of the x "drop off"

PDE : $u_{xx} + u_{yy} = f$

DOM : unit square

BC : $u = g$

TRUE : $e^{(-y^2 - [\frac{\alpha(\beta x)^3}{1+(\beta x)^3}]^2)} + \sin(x-y+\frac{1}{2})$

PARAM : $0 \leq \alpha, \beta \leq 10$

PROBLEM: 18

SOURCE: Artificial

OPER : non-constant coef., analytic functions

B.C. : Dirichlet

SOL : Boundary layer type

PARAM : α, β adjust the shape and strength of x "drop off"

PDE : $u_{xx} + (1+xy)u_{yy} + \cos(x)u_x - e^{-x}uy + 3u = g(x,y)$

DOM : unit square

BC : $u = g$

TRUE : $e^{-y^2 - [\frac{\alpha(\beta x)^3}{1+(\beta x)^3}]^2} + \sin(x-y+\frac{1}{2})$

PARAM : $0 \leq \alpha, \beta, \leq 10$

PROBLEM: 19

SOURCE: Adapted from the nonlinear problem of determining the laminar steady flow of a non-Newtonian fluid

REF : [AMES 65]

OPER : Self-adjoint, non-constant coef. of variable smoothness, singular coef., possible

B.C. : Mixed, Homogeneous

SOL : Analytic used as initial guess for the nonlinear problem

PARAM : α is a physical parameter

PDE : $(wu_x)x + (wu_y)y = f$, $w = [(True_x)^2 + (True_y)^2]^{\alpha}$

DOM : $[.5,1]x[.5,1]$

BC : $\frac{\partial u}{\partial x} = 0$, $x = .5$; $u = 0$, $x = 1.$; $\frac{\partial u}{\partial y} = 0$, $y = .5$; $u = 0$, $y = 1.$

TRUE : $\sin \pi x \sin \pi y$

PARAM : $-\frac{1}{2} \leq \alpha \leq 1$

PROBLEM: 20

SOURCE: Adapted from the nonlinear equation $u_{xx} + u_{yy} = e^u$ which is of interest in diffusion-reaction, vortex problems and electric space charge considerations.

REF : [AMES 65]

OPER : Helmholtz, non-constant coef

B.C. : Dirichlet

SOL. : Peak solution

PARAM : α adjusts coefficient of u and singularity of operator

PDE : $u_{xx} + u_{yy} - wu = f$, $w = \exp(TRUE)$

DOM : $[0,.5]$ x$[0,.25]$

B.C. : $u = g$

TRUE : $10\, \varphi(x)\, \varphi(y) + \alpha$, $\varphi(x) = e^{-100(x-.5)^2}(x^2-x)$

PARAM : $0 \leq \alpha \leq 20$

PROBLEM: 21

SOURCE: Same as problem 20

OPER : Helmholtz, non-constant coef.

B.C. : Dirichlet

SOL : unknown

PARAM : α adjusts coefficient of u and singularity of operator

PDE : $u_{xx} + u_{yy} - wu = (1-p)e^p$, $p = \sin 2\pi x \sin 4\pi y + \alpha$, $w=e^p$

DOM : $[0,15]x[0,.25]$

BC : $u = g$

TRUE : Unknown

PARAM : $0 \leq \alpha \leq 20$

PROBLEM: 22

SOURCE: Adapted from the problem of elastic plastic torsion of a rod with square cross-section of side 2a

REF : [KACHANOV 63]

OPER : Self-adjoint, Discontinuous coef. derivatives

B.C. : Dirichlet

SOL. : Approximate solution computed for the real nonlinear problem, Entire

PARAM : None

PDE : $(\frac{1}{G} u_x)x + (\frac{1}{G} u_y)y = f$, $G = 7.85 \times 10^3$, $\alpha \leq .0025 = 19.4/\alpha + 236 \quad \alpha > .0025$ where $\alpha = ((\text{True}_x)^2 + (\text{True}_y)^2)^{\frac{1}{2}}$

DOM : unit square

B.C. : $u = g$

TRUE : $c_1 F_1 + c_2 F_1(x^2+y^2)$, $F_1=(x^2-1)(y^2-1)$, $c_1 = .0174/1.02 \times 10^{-3}$, $c_2 = -.003693/1.02 \times 10^{-3}$

PARAM : None

PROBLEM: 23

SOURCE: Adapted from the non-linear potential equation for highly saturated electrical machines

REF : [TRUTT 63]

OPER : Self-adjoint, non-constant entire

B.C. : Mixed

SOL. : Wave front in x-direction

PARAM : a and b are physical parameters, the function $\mu(x,y)$ reflects different physical situations

PDE : $(\mu\, u_x)x + (\mu\, u_y)y = f$, $H = ((True_x)^2 + (True_y)^2)^{\frac{1}{2}}$

DOM : unit square

B.C. : $u_x = 0$, $x = 0,1$; $v = g$, $y = 0,1$

TRUE : $\cos\pi x(\varphi(y)+1)$, φ defined in Problem 8

PARAM : The values $a = 387.75$, $b = 50.$ and $a = 554.5$, $b = .544$ have been used previously for $\mu = 1/(a+bH)$, $\mu = H^{-1}\exp(H/(a+bH))$ and $\mu = aH^{-1}\tanh(bH)$

PROBLEM: 24

SOURCE: Adapted from real problem

REF : [BEHNKE 68]

OPER : Non-constant coef., Possible singular coefficients

B.C. : Dirichlet, homogeneous

SOL. : Unknown

PARAM : R, C and β are physical parameters. R adjusts singularity of operator, C is the forcing term and β changes the size of the domain.

PDE : $u_{xx} + u_{yy} - \frac{2}{R-y} u_y = C$

DOM : $[0,\beta]x[0,\beta]$

B.C. : $u = 0$

TRUE : Unknown

PARAM : We have R = 1/250, C = 1, and β = 1 or 10

PROBLEM: 25

SOURCE: Adapted from real problem

REF : [BEHNKE 68]

OPER : Non-constant coefficients, Oscillatory

B.C. : Mixed boundary cond ., Small coeff., Homogeneous

SOL. : Unknown

PARAM : C,h(x,y), g(x,y), α and β are physical parameters and functions.

PDE : $u_{xx} + u_{yy} + \frac{3}{h} u_x h_x = c\, h_x/h^3$

DOM : unit square

BC : $u + \alpha u_x + \beta u_y = g$

TRUE : Unknown

PARAM : Values to be used are: c=1; h = sin(πxy) or h=sin(100πxy); g(x)=0 and α=β=1/6 for ε = 1,.1,.01,.001.

PROBLEM: 26

SOURCE: Adapted from real problem

REF : [BEHNKE 68]

OPER : Non-constant coef.

B.C. : Dirichlet, homogeneous

SOL. : Unknown

PARAM : ℓ is a physical parameter

PDE : $u_{xx} + u_{yy} + \frac{6x(1+x^2)^2}{(\ell+x^2)^3} u_x = \frac{12x}{(\ell+x^2)^3}$

DOM : $[0,\ell]^2$

BC : $u = 0$

TRUE : Unknown

PARAM : $1 \leq \ell \leq 10$

PROBLEM: 27

SOURCE: Adapted from real problem

REF : [BEHNKE 68]

OPER : Non-constant, singular coefficients at the boundary

B.C. : Dirichlet, homogeneous

SOL. : Unknown

PARAM : C is a forcing function

PDE : $u_{xx} + \frac{2}{x} u_x + \frac{1}{x^2} u_{yy} + \frac{1}{x^2} \cot^3 y \, u_y = C$

DOM : $[0,1]x[0,1]$

BC : $u = 0$

TRUE : Unknown

PARAM : C = 1 has been used previously.

PROBLEM: 28

SOURCE: Adapted from the balance equation in meteorology

REF : [FOX 62]

OPER : Non-constant coeff., cross derivative term

B.C. : Dirichlet, homogeneous

SOL. : Unknown

PARAM : β adjusts the size of the domain.

PDE : $p\,\psi_{xx} + g\psi_{xy} + v\psi_{yy} + s\psi_x + t\psi_y = g$

$p = f$, $v = -2(\text{True})_{xy}$, $g = f + 2(\text{True})_{xx}$

$s = f_x$, $t = f_y$ where $f = e^{x+y}$

DOM : $[0,\beta]x[0,\beta]$

BC : $u = g$

TRUE : Unknown

PARAM : β = 1,10 and 100 have been used previously.

REF L. Fox, 'Numerical Solution of Ordinary and Partial Differential Equations', Pergamon Press 1962.

PROBLEM: 29

SOURCE: K = 1 Potential of electric, magnetic and gravitational fields, steady flow of heat, electric current and ideal fluids

K =-1 Streamlines of steady fluid flow

K = 3 Torsion of a shaft

K =-3,5 Stress concentration theory

REF : [GREENS 68]

OPER : Singular coefficients

B.C. : Dirichlet

SOL. : Unknown

PARAM : K is a physical parameter

PDE : $u_{xx} + u_{yy} + \frac{K}{y} u_y = 0$

DOM : Unit square

BC : $u = x - y$

TRUE : Unknown

PARAM : K = ± 1, ± 3, 5 have been used previously.

REF D. Greenspan, 'Lectures on the Numerical Solution of Linear, Singular, and nonlinear differential equations, Prentice-Hall, 1968.

PROBLEM: 30

SOURCE: Artificial

OPER : Variable smoothness coefficients, Possible singularities

B.C. : Dirichlet, homogeneous

SOL. : Analytic

PARAM : α adjusts smoothness of operator coefficients without affecting solution.

PDE : $-(x^{\alpha}u_x) - (y^{\alpha}u_y)y + (xy)^{\alpha}u = f$

DOM : unit square

BC : $u = 0$

TRUE : $3e^{x+y}(x-x^2)(y-y^2)$

PARAM : $\frac{1}{2} \leq \alpha \leq 5$

PROBLEM: 31

SOURCE: Temperature distribution

REF : [COLLATZ 66]

OPER : Poisson

B.C. : Mixed

SOL. : Analytic approximation solution with $u_n = -u$ boundary

PARAM : None

PDE : $u_{xx} + u_{yy} = -1$

DOM : $[-1,1]x[-1,1]$

BC : $\frac{\partial u}{\partial \nu} + u = g$

TRUE : $-\frac{1}{4}(x^2+y^2) + \alpha_0 + \alpha_2(x^4 - 6x^2y^2 + y^4) +$
$\alpha_4(x^8 - 28x^6y^2 + 70x^4y^4 - 28x^2y^6 + y^8) +$
$\alpha_6(x^{12} - 66x^{10}y^2 + 495x^8y^4 - 924x^6y^6 + 495x^4y^8$
$- 66x^2y^{10} + y^{12})$

$\alpha_0 = .821564$, $\alpha_2 = 0.01440$, $\alpha_4 = .0000493$, $\alpha_6 = -.00000064$

PARAM : None

PROBLEM: 32

SOURCE: Shear stress in helical spring

REF : [COLLATZ 66]

OPER : Possible singular coefficient;

B.C. : Dirichlet, Homogeneous

SOL. : Entire, Ritz method approximation for R = 5, $2G\lambda = 1$

PARAM : R, G, α are physical parameters

PDE : $u_{xx} + u_{yy} + \frac{3}{R-y} u_y - 2G\lambda = f$, $\lambda = \sin\alpha\cos\alpha/R$

DOM : $[-.5,.5]x[-1.,1.]$

B.C. : $u = 0$

TRUE : unknown in general, $(1-y^2)(1-4x^2)(5-y)^3(a+by)$, $a = .0010185$,

$b = .0004838$ for R = 5, $2G\lambda = 1$

PARAM : $0 \leq \alpha \leq \pi/2$, $2 \leq R \leq 10$, $1 \leq G \leq 10^6$

PROBLEM: 33

SOURCE: Torsion problem for a shaft of square cross-section

REF : [COLLATZ 66]

OPER : Poisson

B.C. : Dirichlet

SOL. : Entire Power Series expansion for f = -1 and g = 0.

PARAM : None

PDE : $u_{xx} + u_{yy} = f$

DOM : $[-1,1]x[-1,1]$

BC : $u = g$

TRUE : $C - \frac{1}{4}(x^2+y^2) - \frac{14476}{319424}(x^4 - 6x^2y^2 + y^4) +$

$\frac{429}{319424}(x^8 - 28x^6y^2 + 70x^4y^4 - 28x^2y^6 + y^8)$

$C = .295776$

PARAM : None

PROBLEM: 34

SOURCE: Adapted from a problem with infinite region

REF : [COLLATZ 66]

OPER : Poisson

B.C. : Dirichlet

SOL. : Entire

PARAM : None

PDE : $u_{xx} + u_{yy} = f$

DOM : $[0,1]x[-1,1]$

BC : $u = g$

TRUE : $(1-y^2)f_1(x) + (1-y^2)f_2(x)$, $f_1(x) = \alpha e^{k_1 x} + (1-\alpha)\, e^{k_2 x}$

$f_2(x) = be^{k_1 x} - b\, e^{k_2 x}$, $k_1 = \sqrt{p_1}$, $k_2 = \sqrt{p_2}$, $p_1 = 14 + \sqrt{133}$,

$p_2 = 14 - \sqrt{133}$, $a = \dfrac{7-p_2}{2\sqrt{133}}$, $b = \dfrac{7-p_1}{16}\, a$

PROBLEM: 35

SOURCE: Torsion problem for a beam of square cross-section

REF : [COLLATZ 66]

OPER : Laplace

B.C : Dirichlet

SOL. : Analytic approximate solution for a physical problem with boundary condition $u = 0$.

PARAM : None

PDE : $u_{xx} + u_{yy} = 0$

DOM : $[-1,1]x[-1,1]$

BC : $u = g$

TRUE : $u_1 = x^4 - 6x^2y^2 + y^4$

$u_2 = x^8 - 28x^6y^2 + 70x^4y^4 - 28x^2y^6 + y^8$

$u = 1.1786 - .2\, u_1 + .006\, (u_2 + 3\, u_1) + .0019\, u_1$

PROBLEM: 36

SOURCE: Adapted from problem 27. The coefficient of u_y term was modified.

REF :

OPER : Non-constant, singular coefficients at the boundary

B.C. : Dirichlet

SOL. : Entire

PARAM : None

PDE : $u_{xx} + \frac{2}{x} u_x + \frac{1}{x^2} u_{yy} + \frac{1}{x} \cot y \, u_y = C$

DOM : $[0,1]x[0,1]$

B.C : u = True

TRUE : e^{x+y}

PARAM : None

PROBLEM: 37

SOURCE: Adapted from the nonlinear minimal surface equations

REF : [AMES 65].

OPER : Non-constant coefficients, u_{xy} term

B.C. : Dirichlet

SOL. : Entire

PARAM : None

PDE : $au_{xx} + bu_{xy} + cu_{yy} = f$ where $a = 1 + (True_y)^2$,

$b = -2\,(True_x)(True_y)$,

$c = 1 + (True_x)^2$

DOM : unit square

B.C. : u = True

TRUE : $(x-3y)^4$

PARAM : None

PROBLEM: 38

SOURCE: Artificial

REF :

OPER : Poisson, Right-side function of $x^{\alpha/2-2}$, $y^{\alpha/2-2}$

Singular for $\alpha \leq 3$

B.C. : Dirichlet

SOL. : Function of $x^{\alpha/2}$, $y^{\alpha/2}$

PARAM : α adjusts singularity strength

PDE : $u_{xx} + u_{yy} = f$

DOM : unit square

B.C. : u = True

TRUE : $(xy)^{\alpha/2}$

PARAM : $1.0 < \alpha < 5.0$

PROBLEM: 39

SOURCE: Artificial

REF :

OPER : Non-constant coefficients

B.C. : Dirichlet

SOL. : Entire - to singular

PARAM : a,b,c adjust the smoothness of the solution and operator .

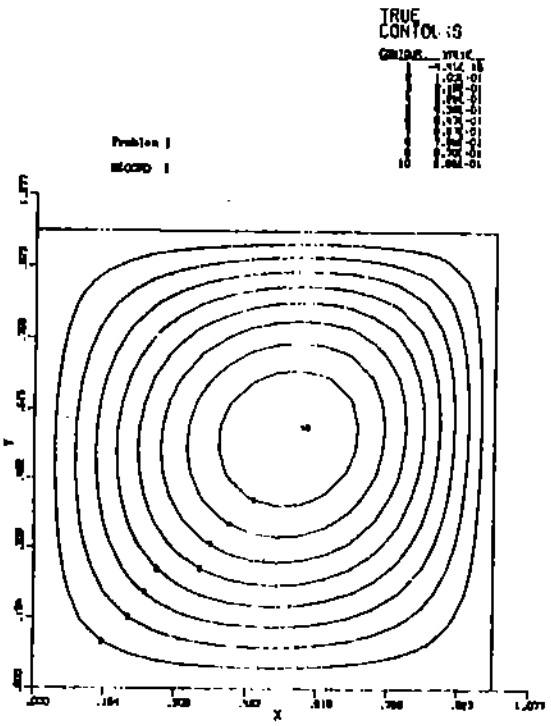
PDE : $(2 + (y-1)\, e^{-ay^4})\, u_{xx} + (1+1/(1+(2x)^b))u_{yy} + c(x(x-1) + (y-.3)(y-.7)u = f$

DOM : unit square

B.C : u = True

TRUE : $(x+y^2)/(1+(2x)^{b-1}) + (y-1)(1+x)e^{-ay^4} + c(x+y)\cos(xy)$

PARAM : $0 < a \leq 100$, $1 < b < 6$, $0 < c \leq 10$.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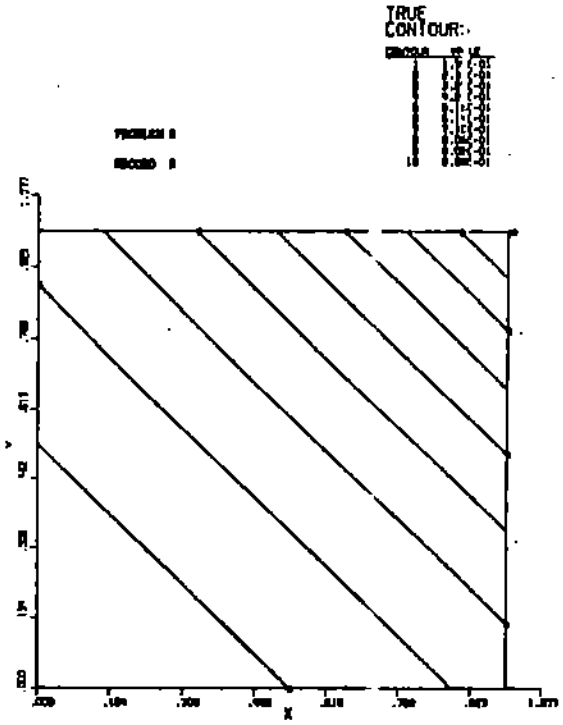
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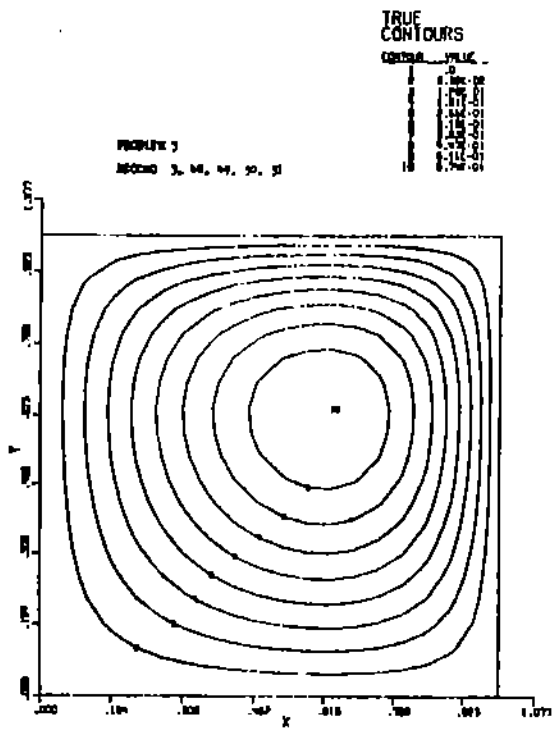
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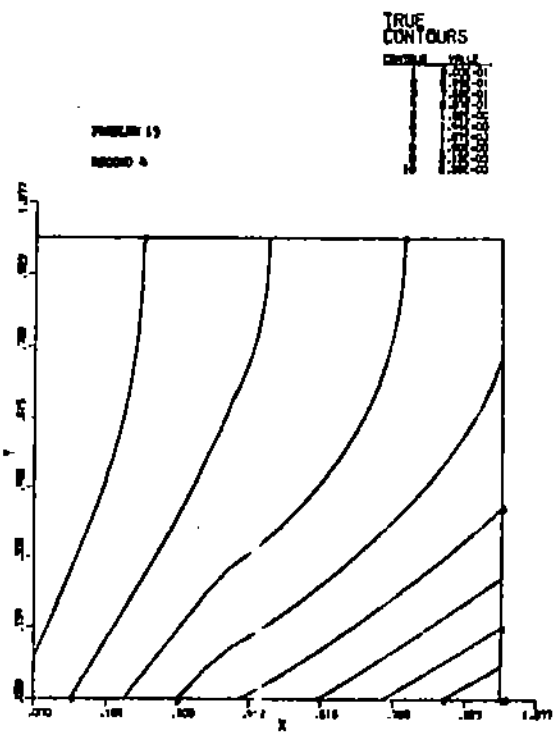
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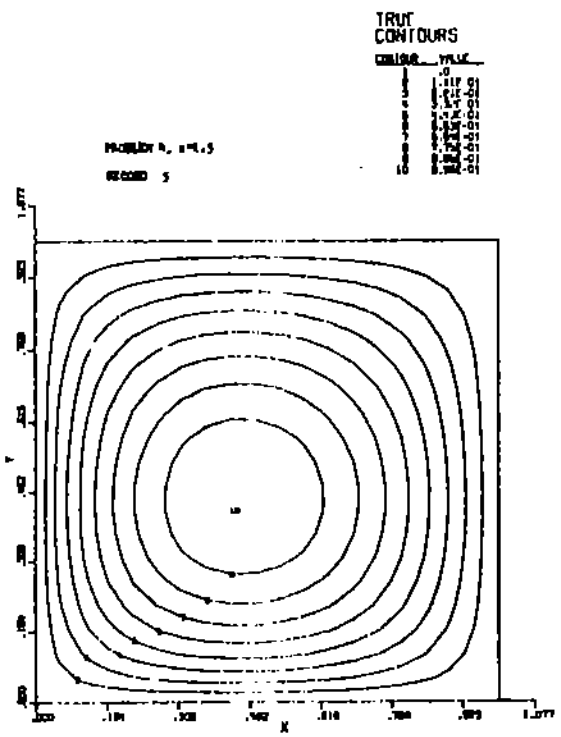
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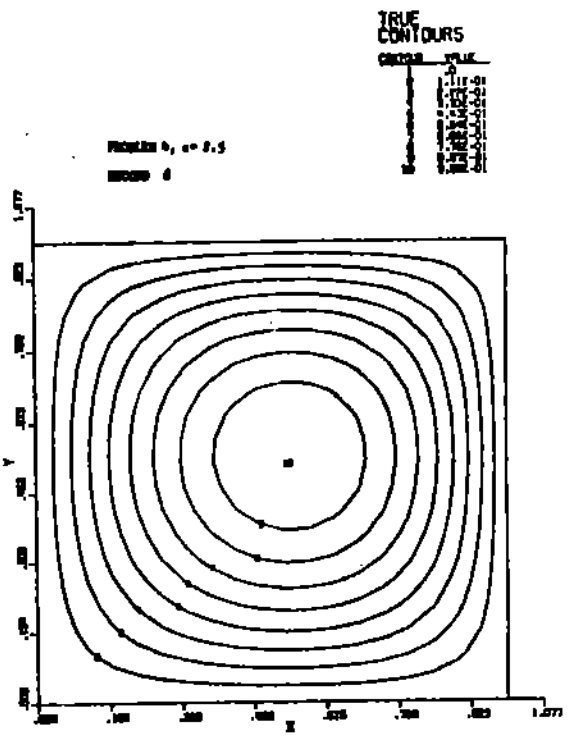
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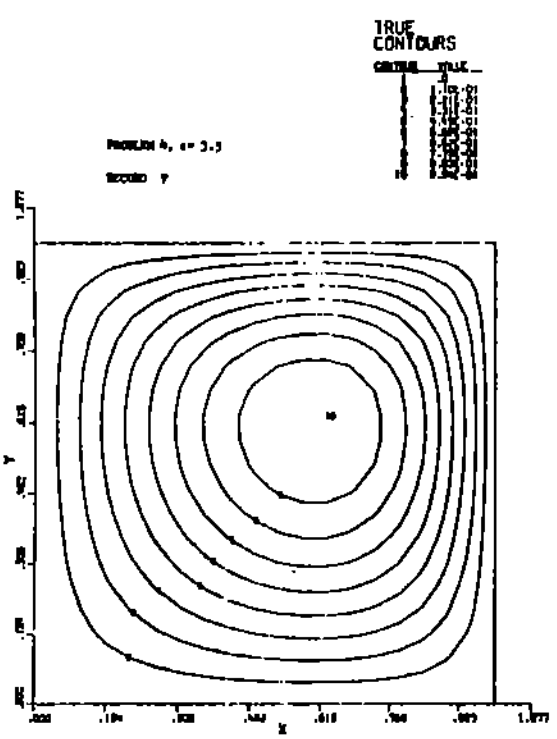
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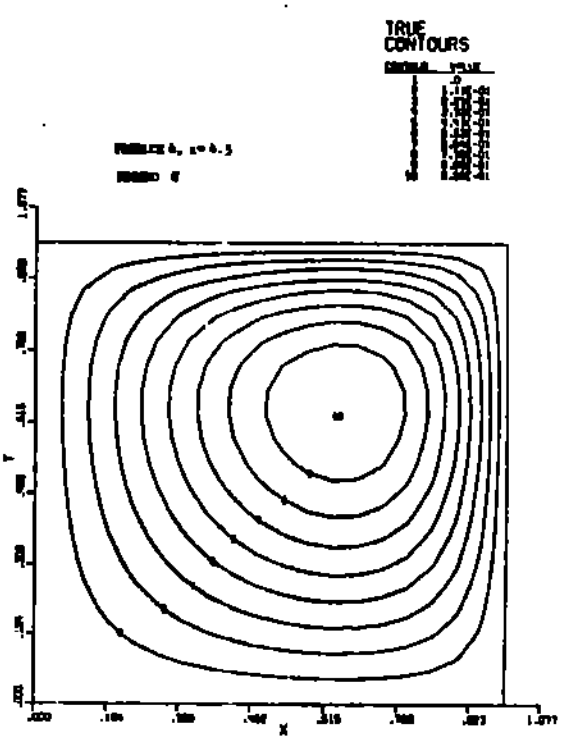
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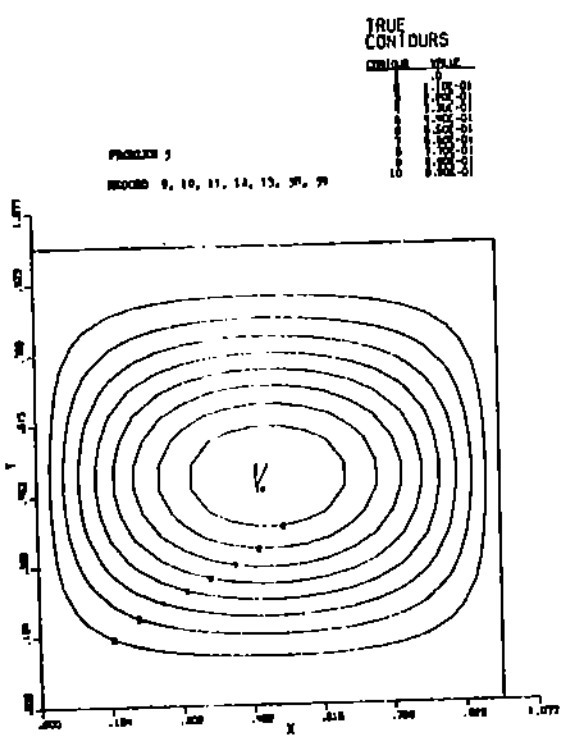
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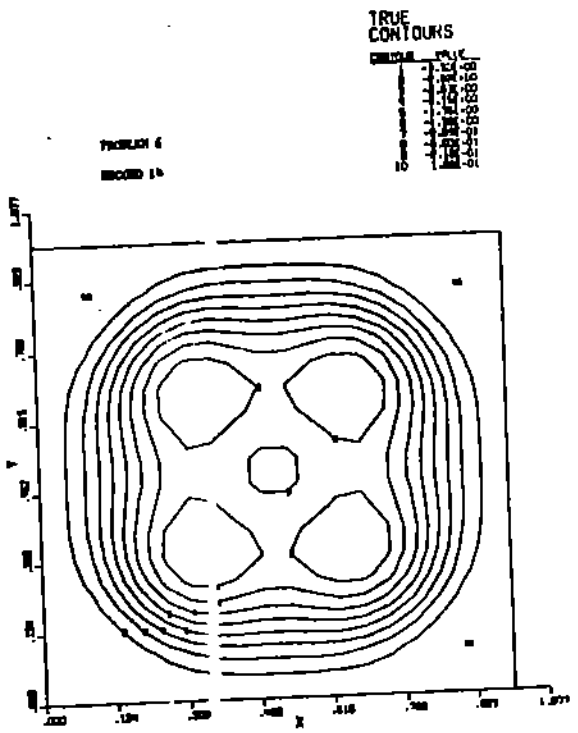
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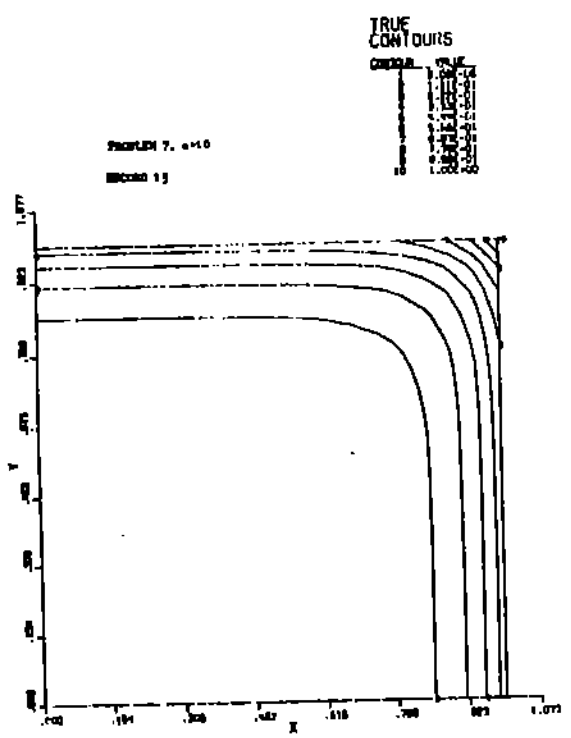
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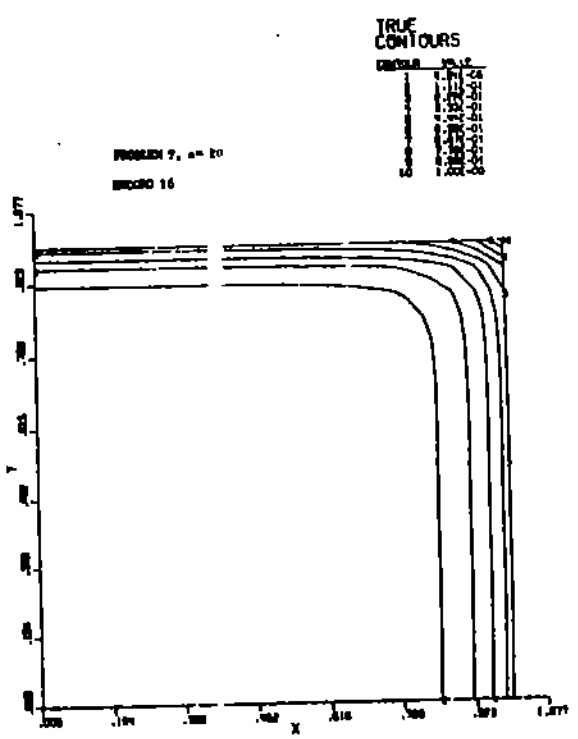
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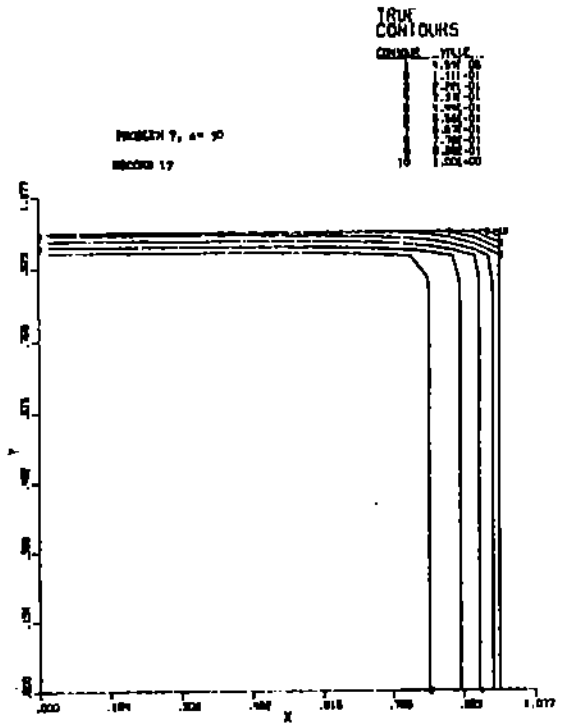
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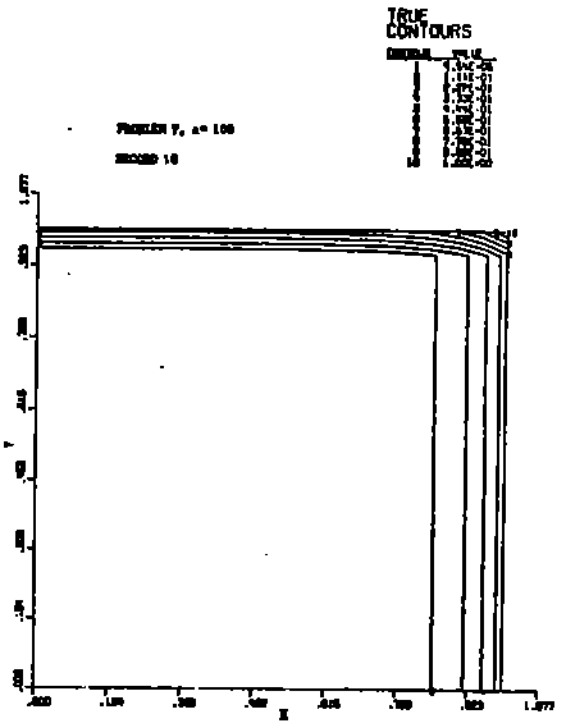
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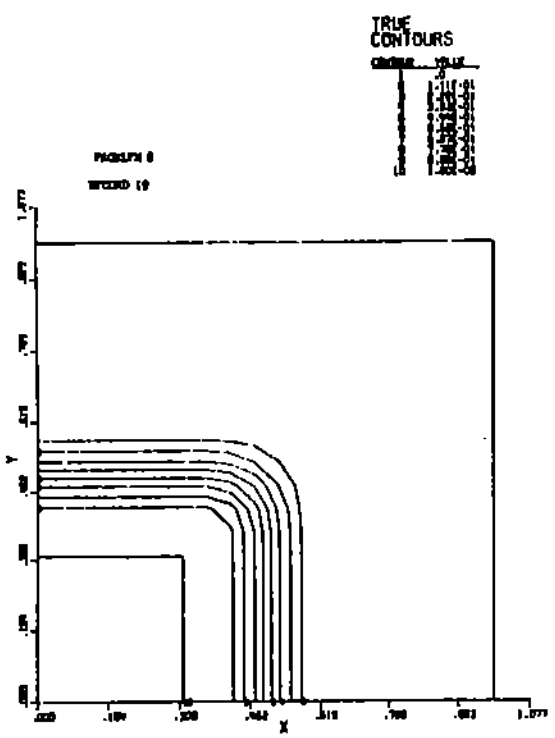
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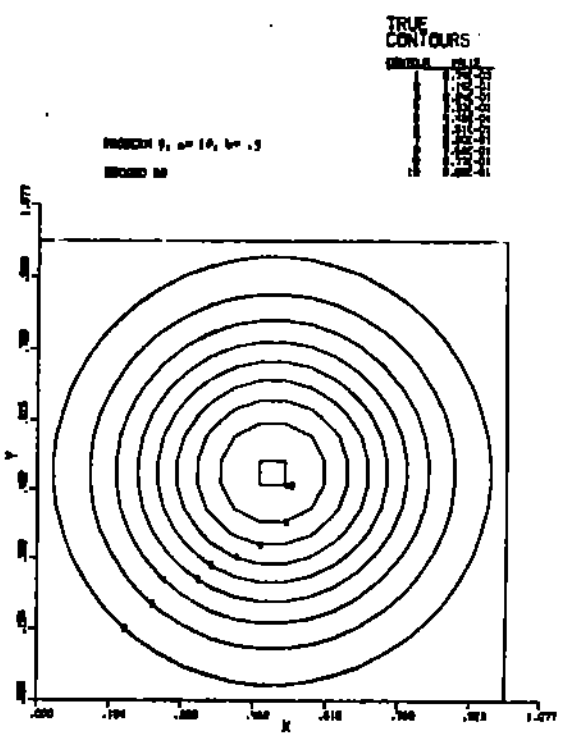
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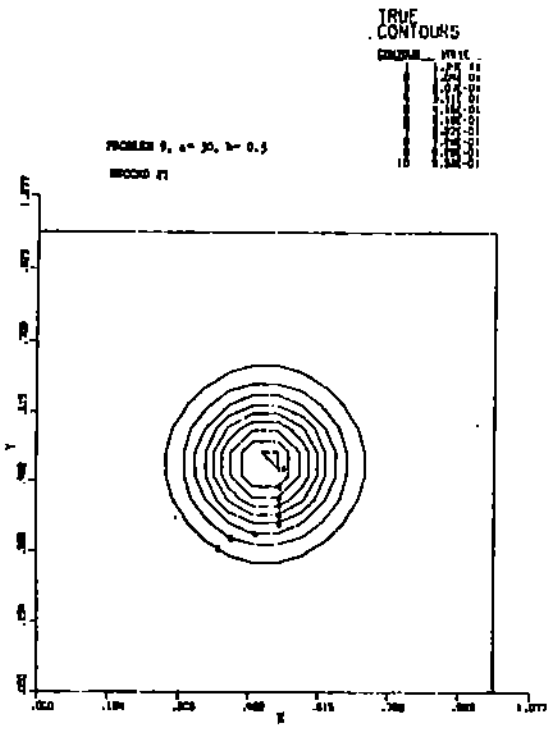
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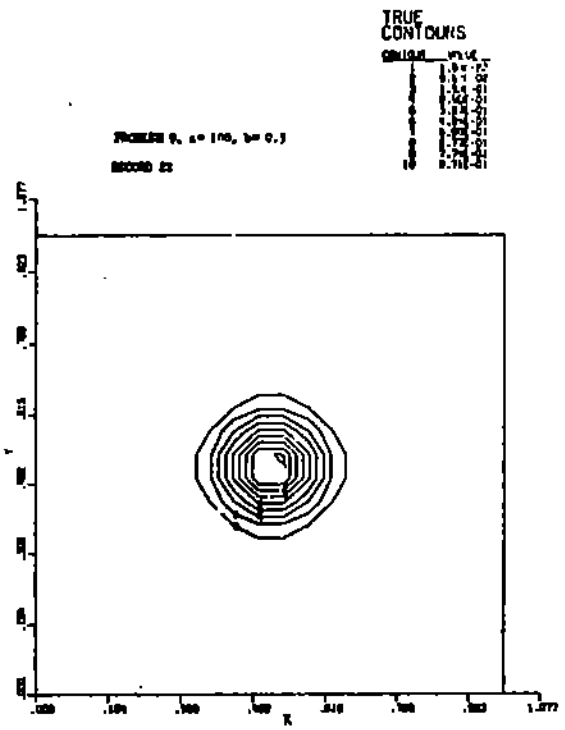
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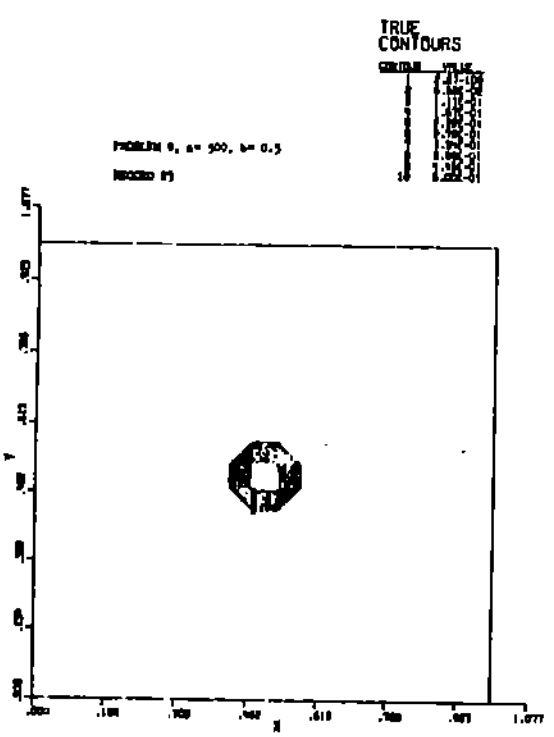
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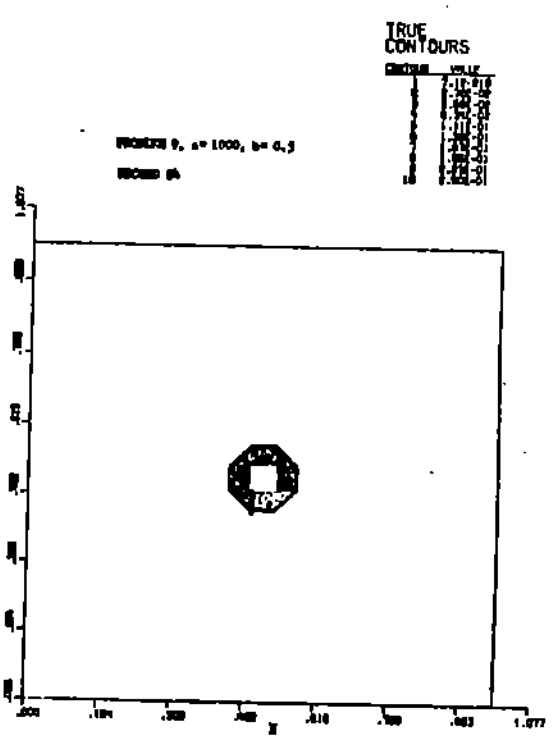
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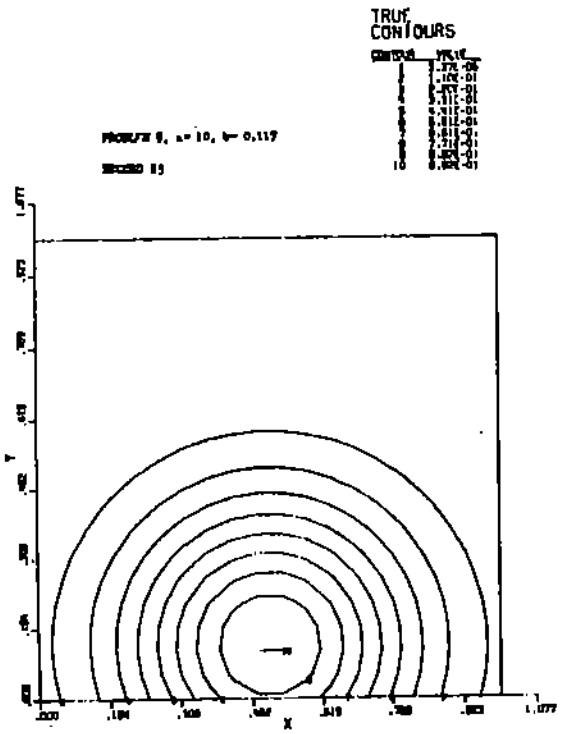
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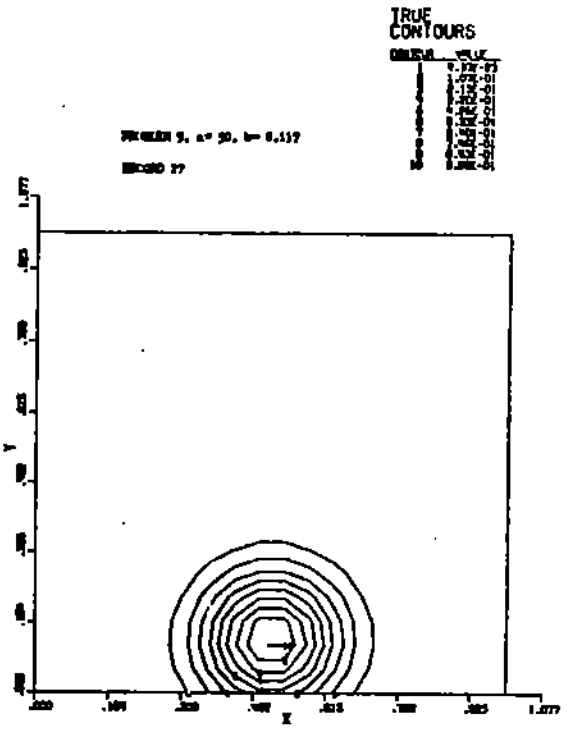
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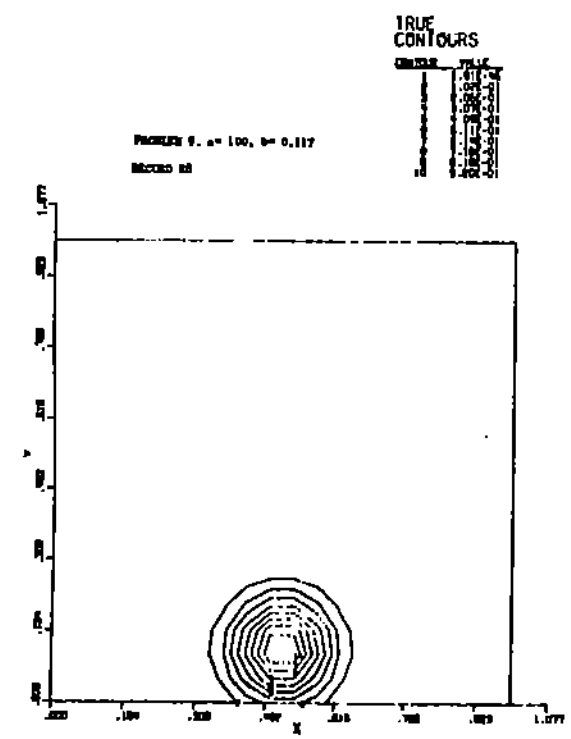
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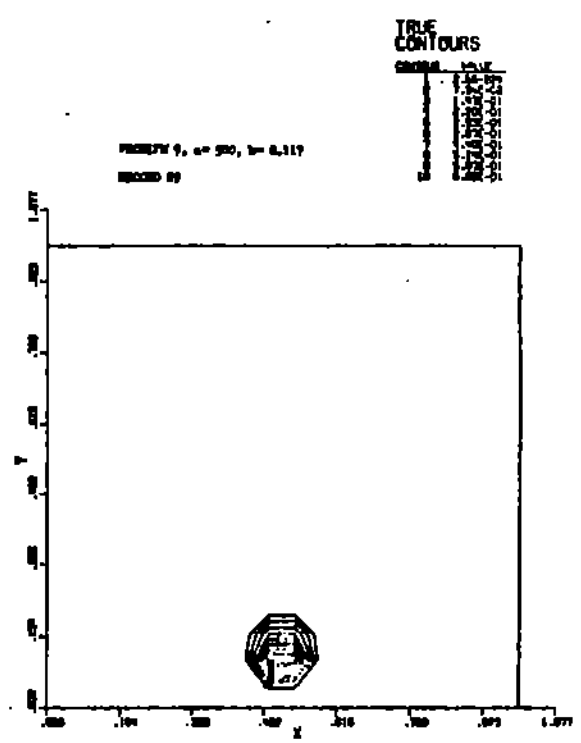
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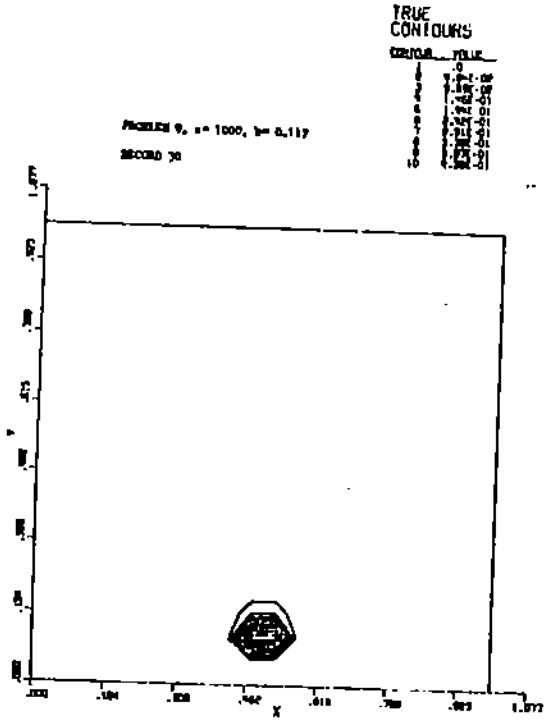
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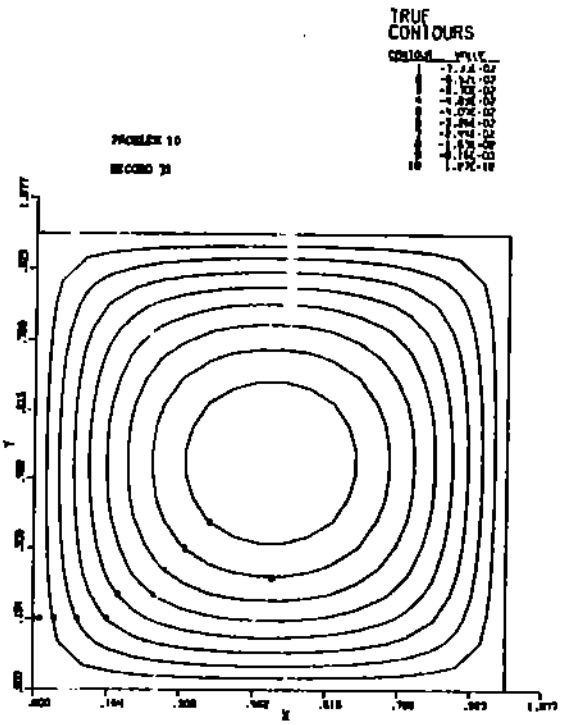
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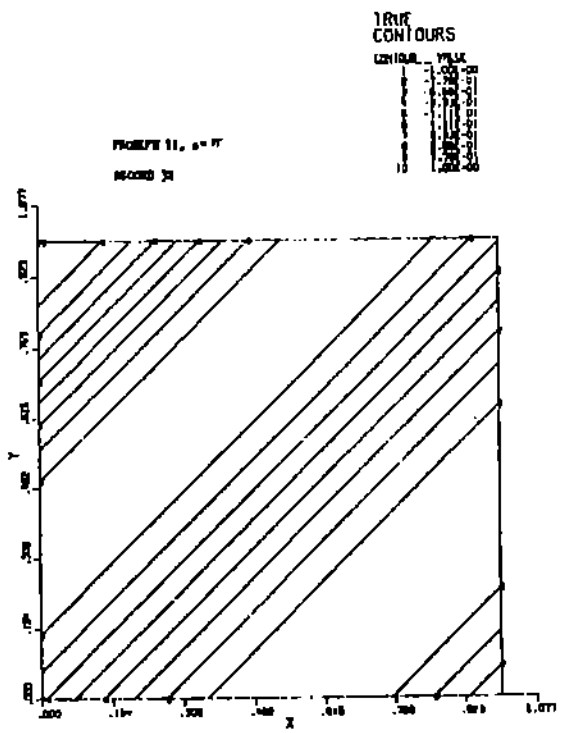
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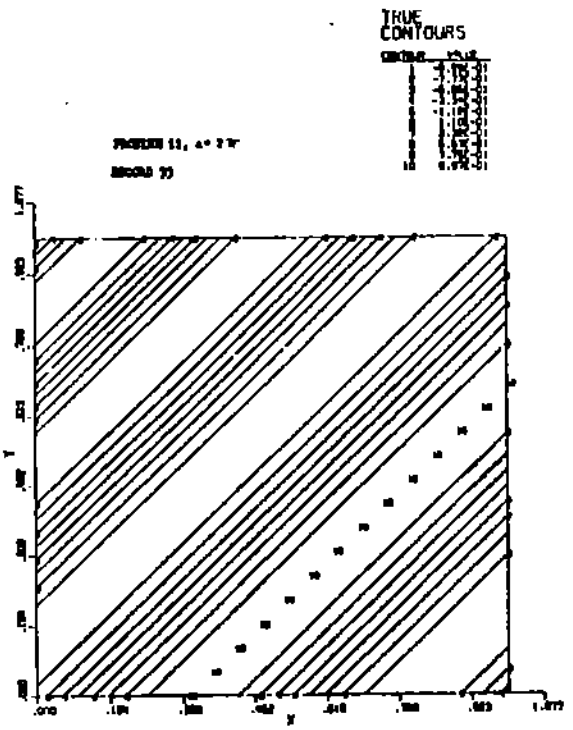
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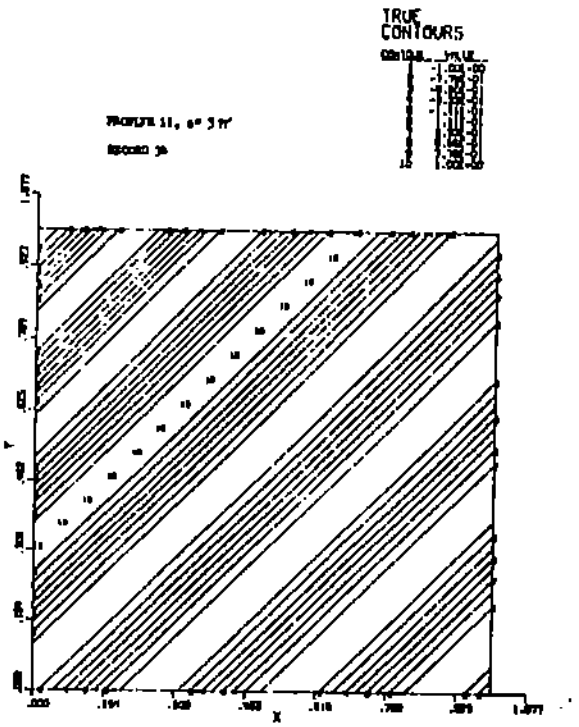
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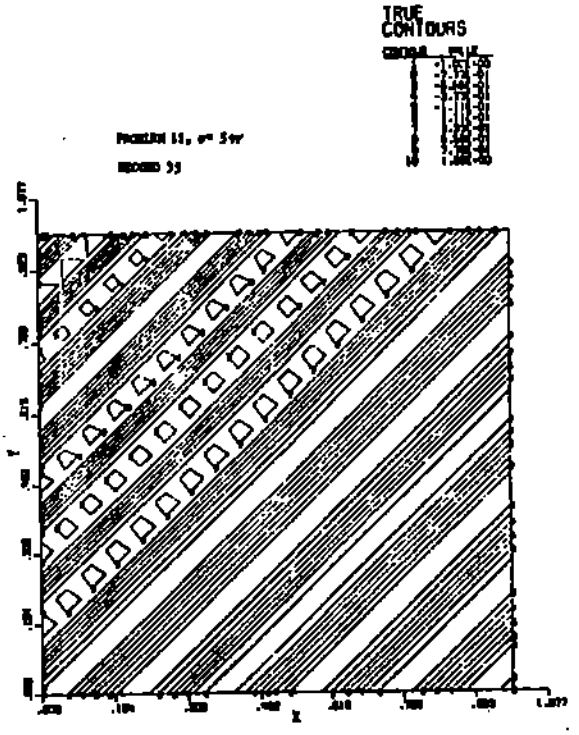
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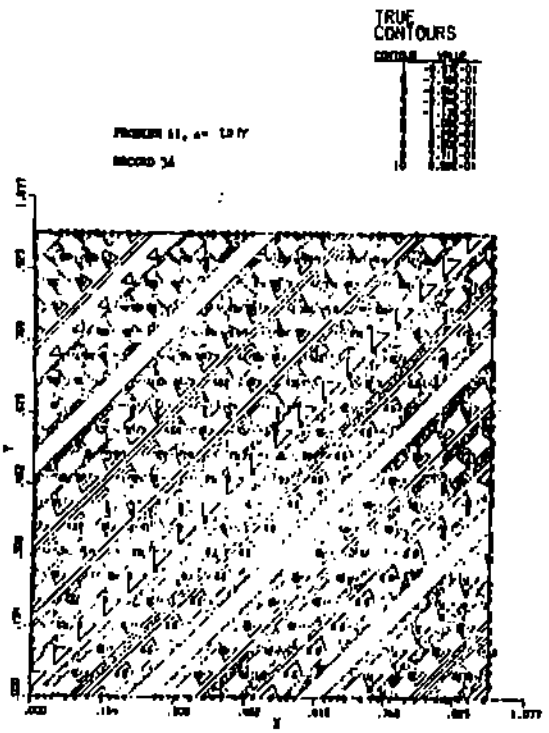
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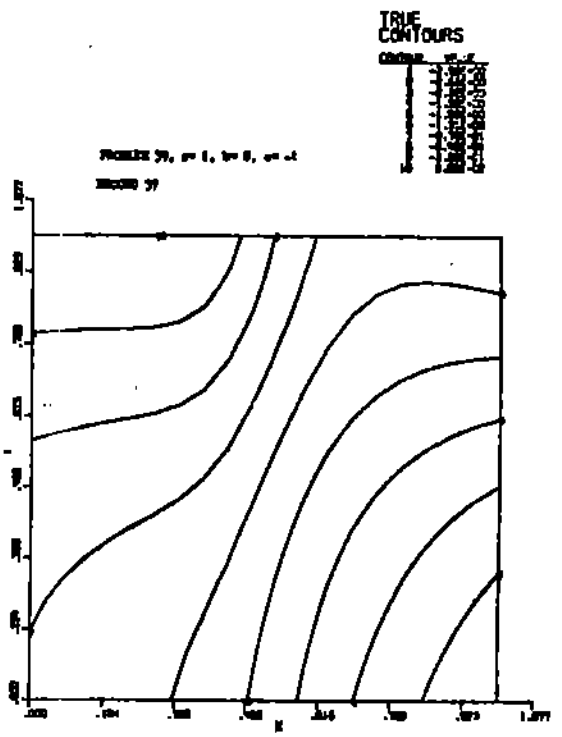
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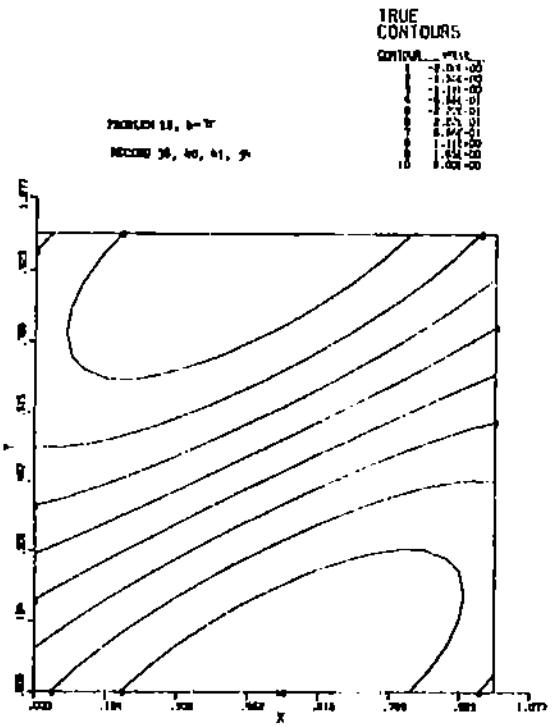
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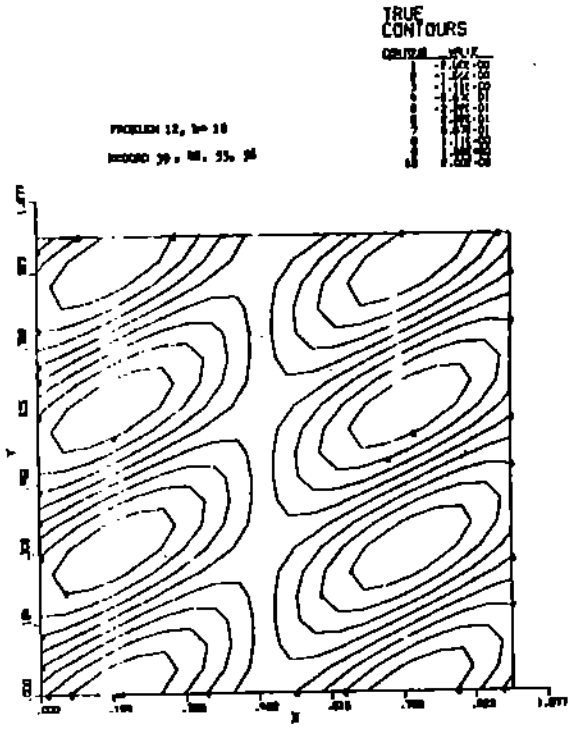
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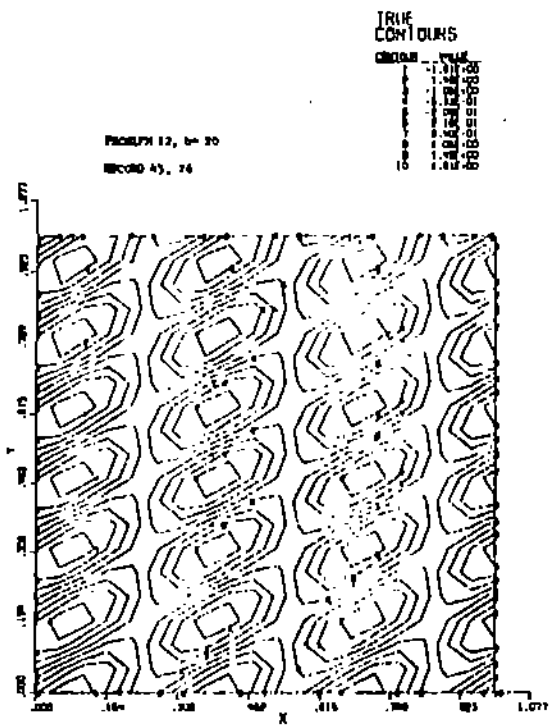
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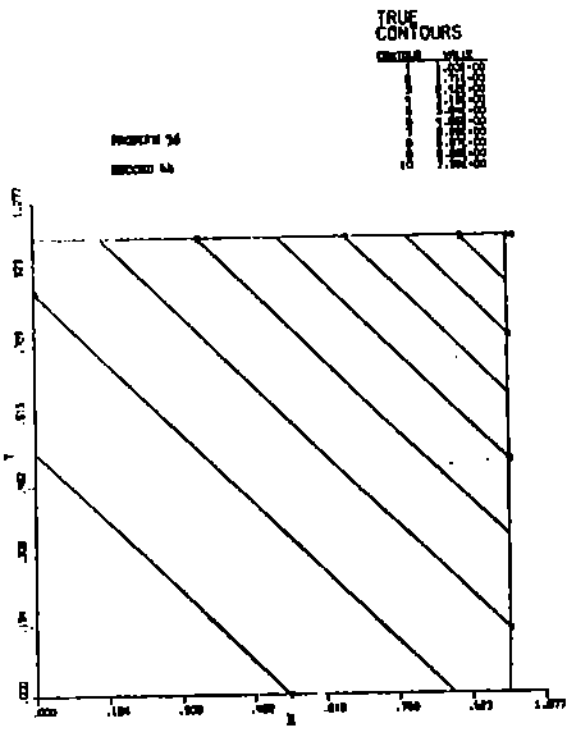
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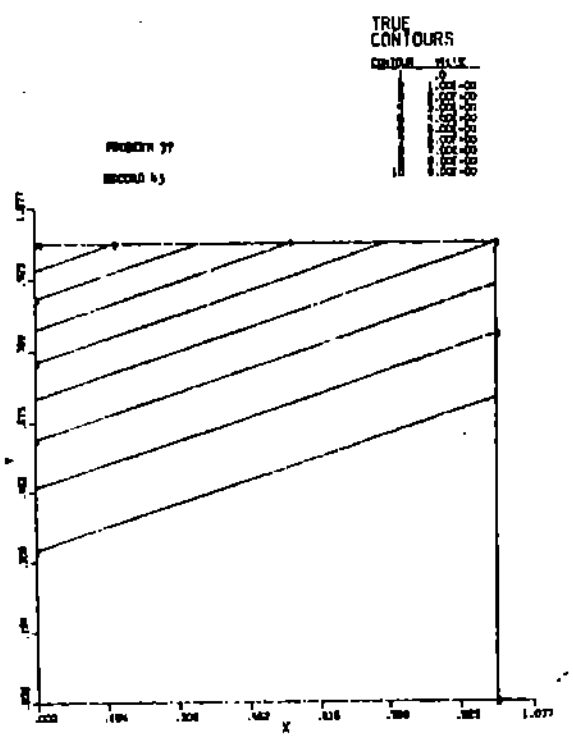
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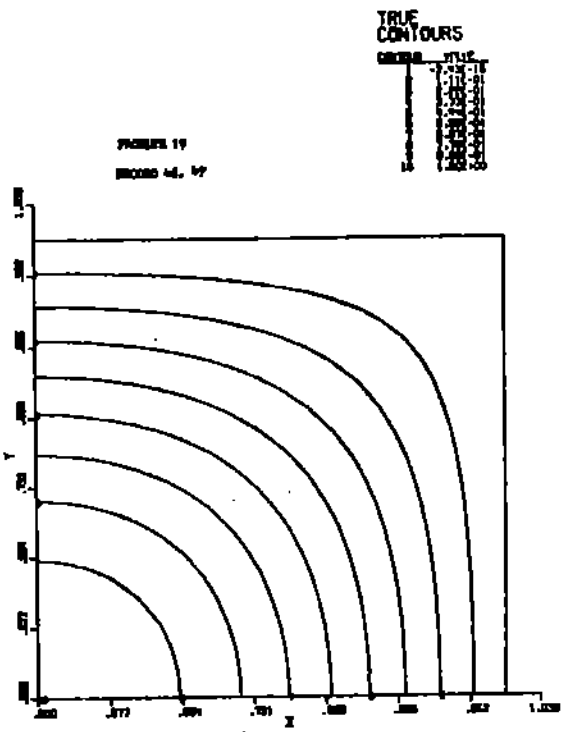
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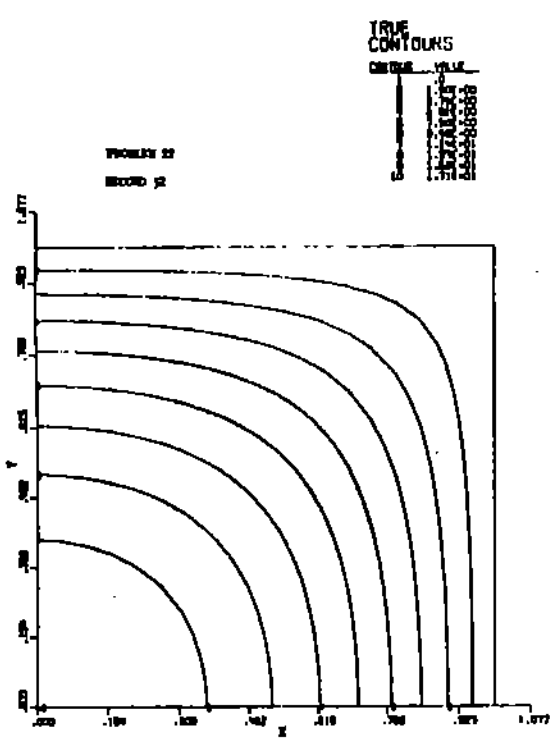
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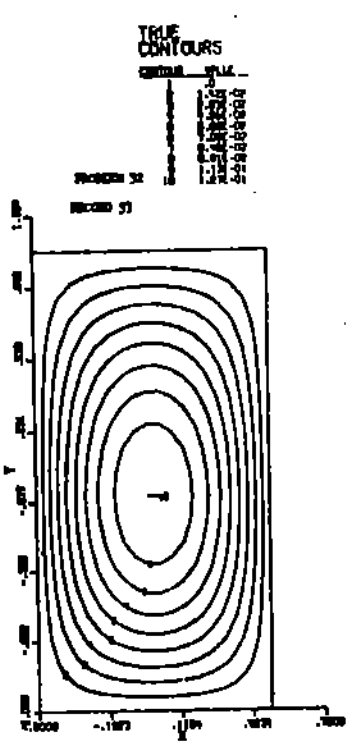
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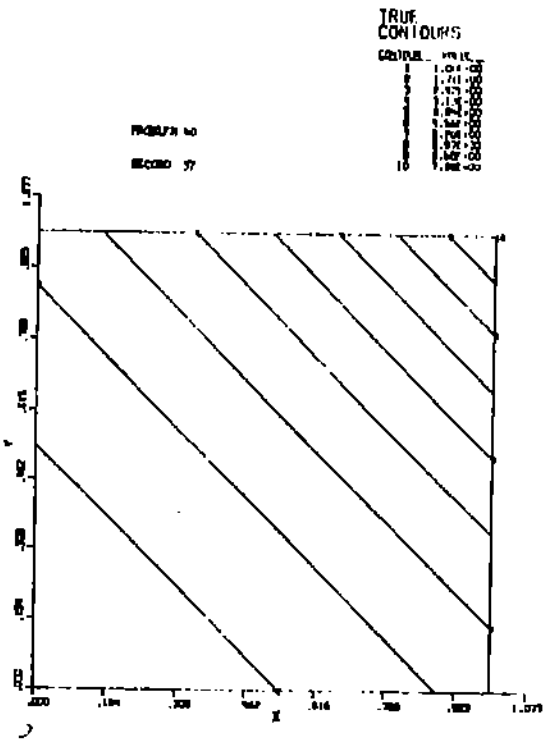
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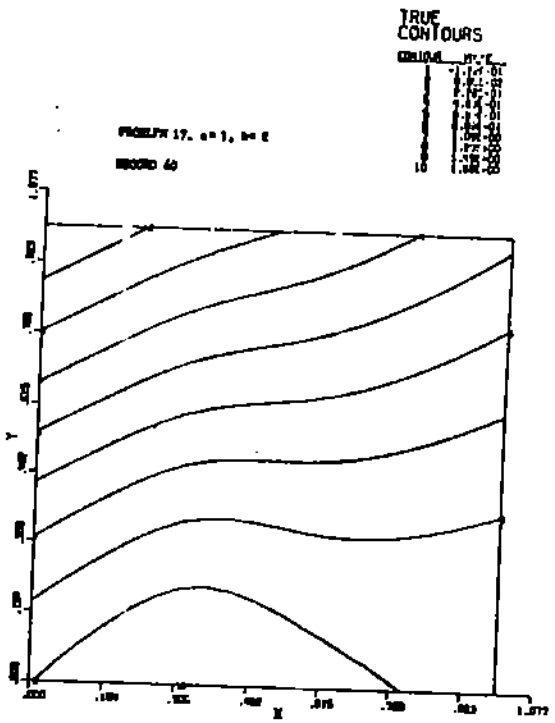
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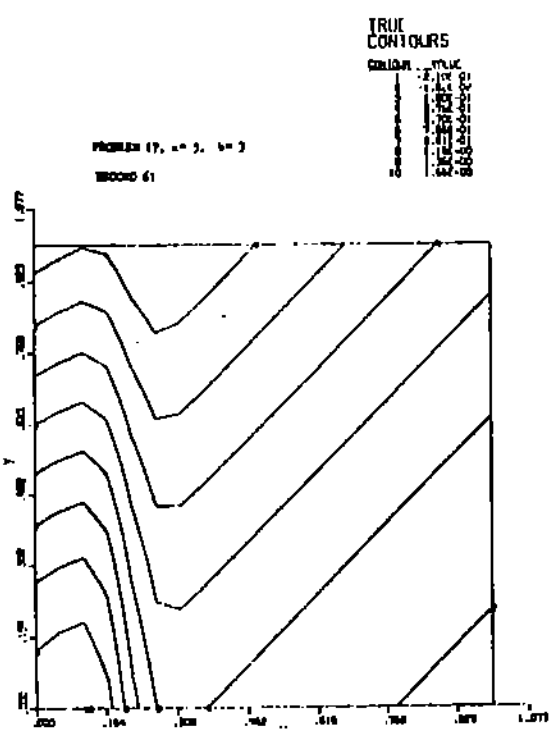
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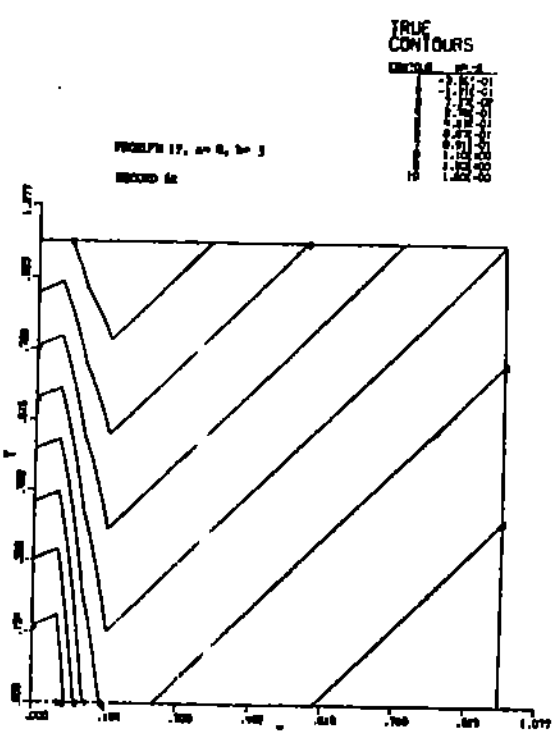
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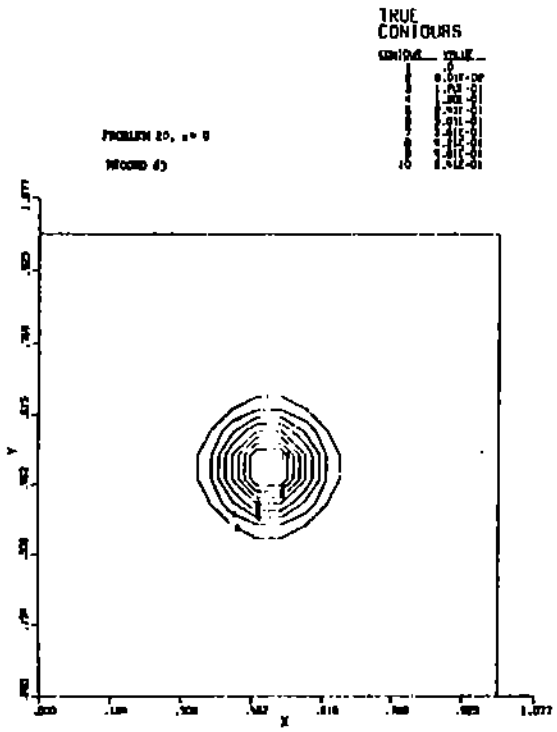
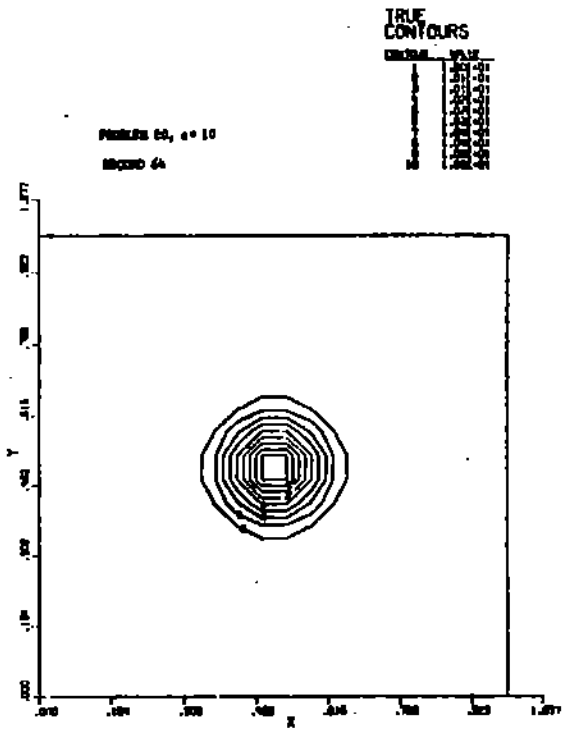
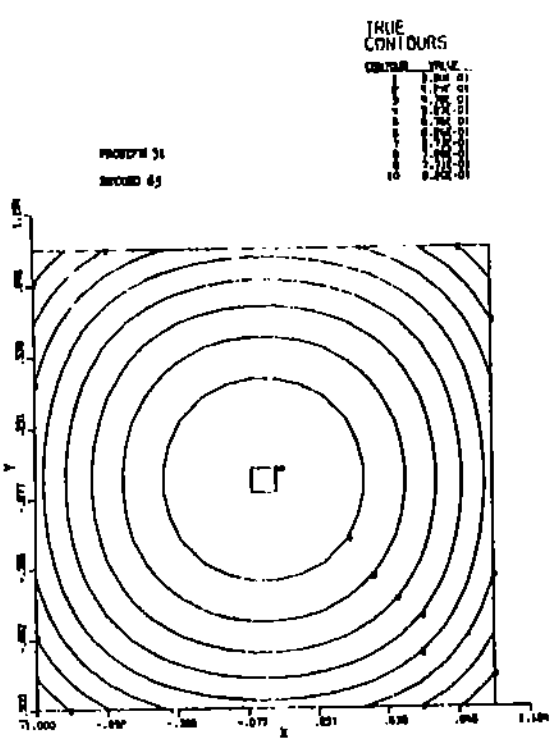
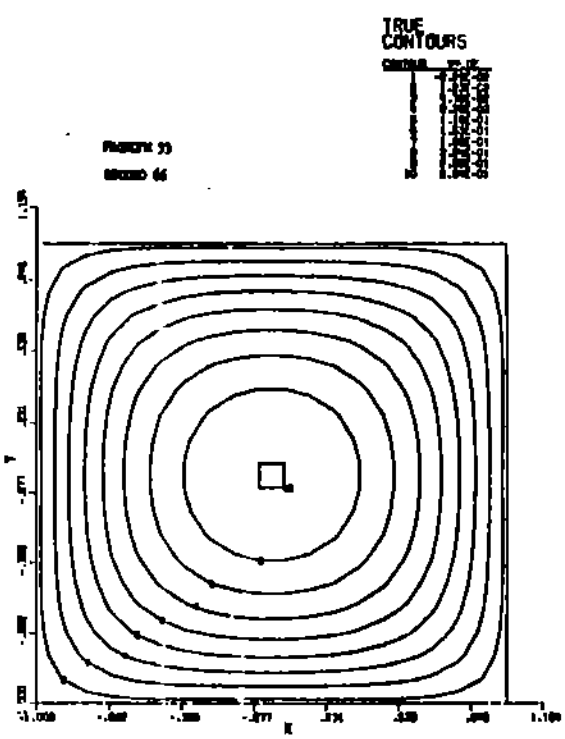
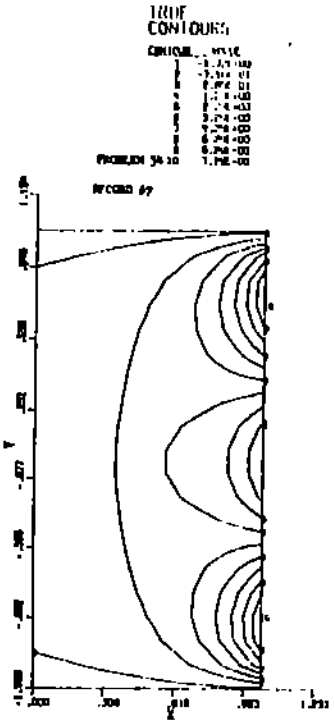
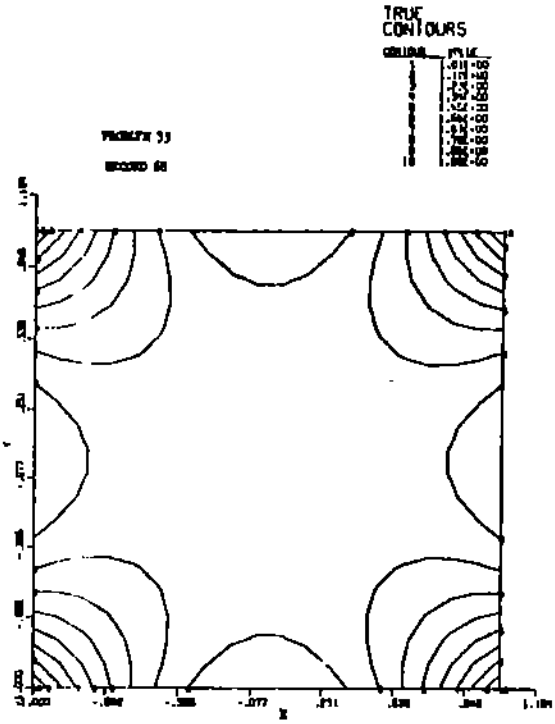

TRUE
CONTOURS

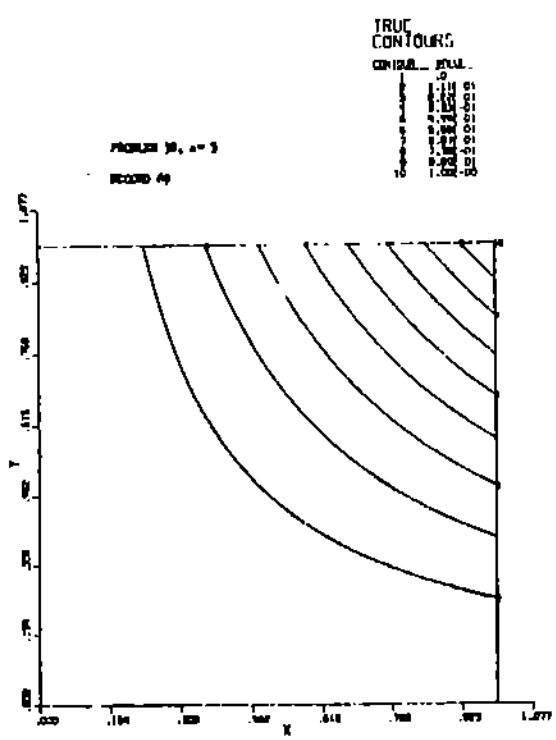

TRUE
CONTOURS

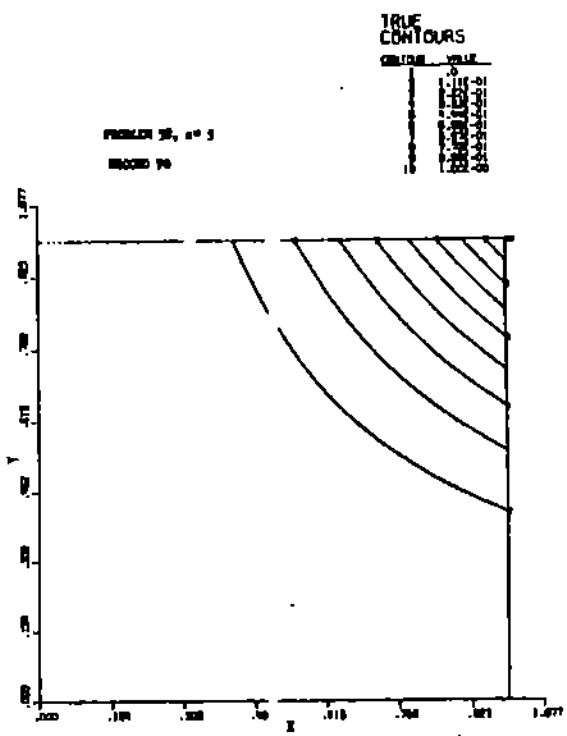

TRUE
CONTOURS

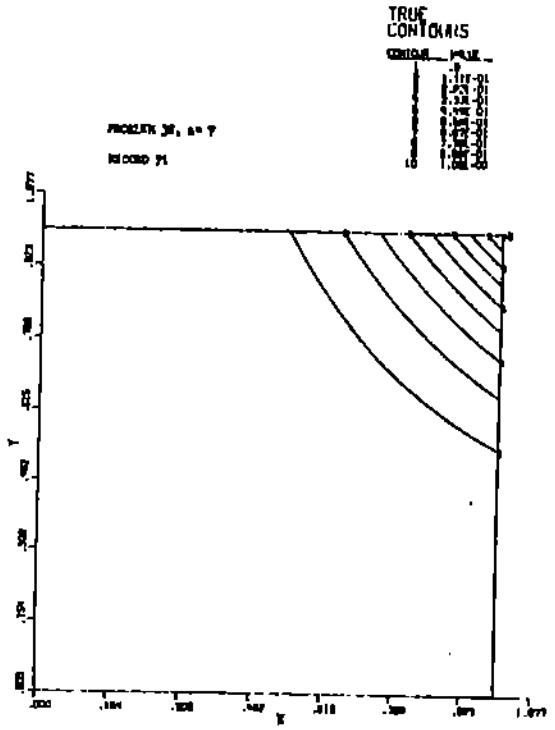

TRUE
CONTOURS

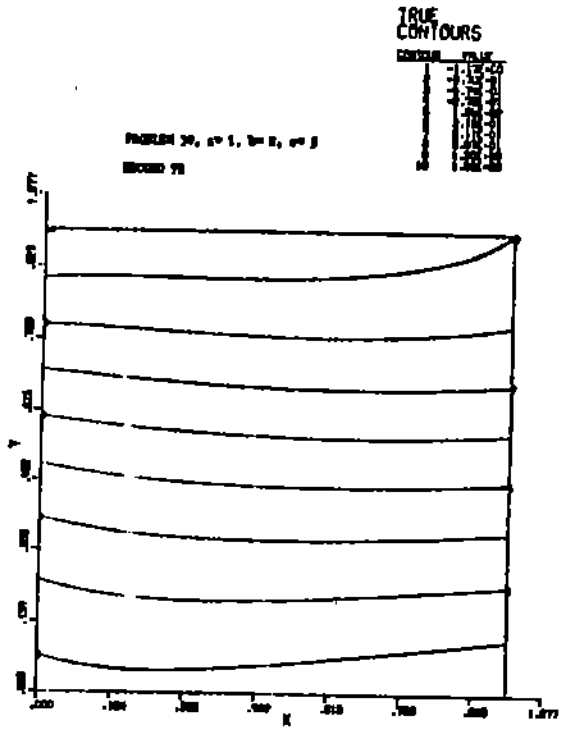

TRUE
CONTOURS

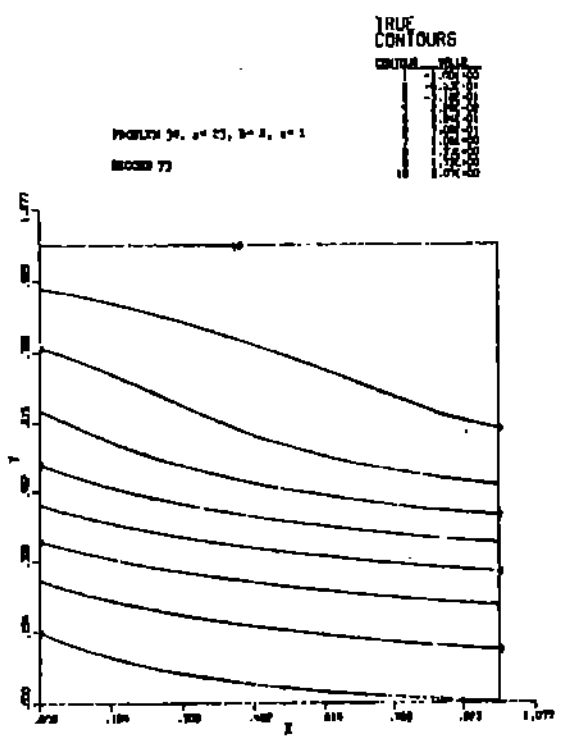

TRUE
CONTOURS

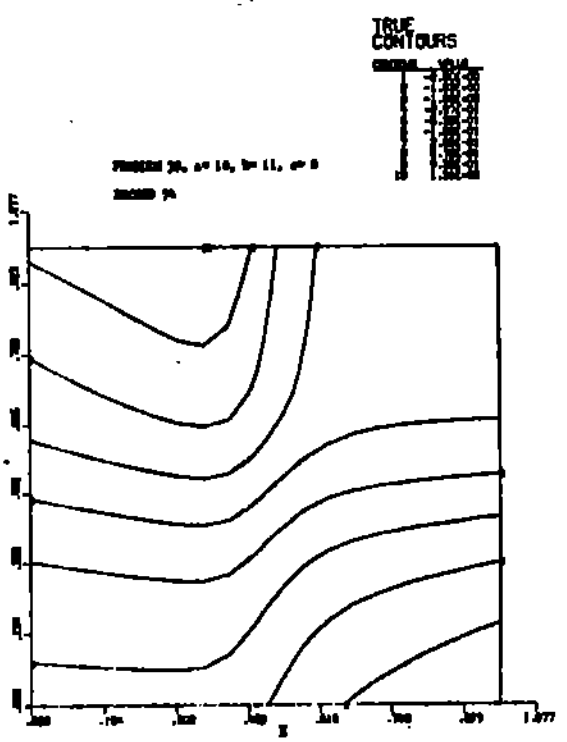

TRUE
CONTOURS

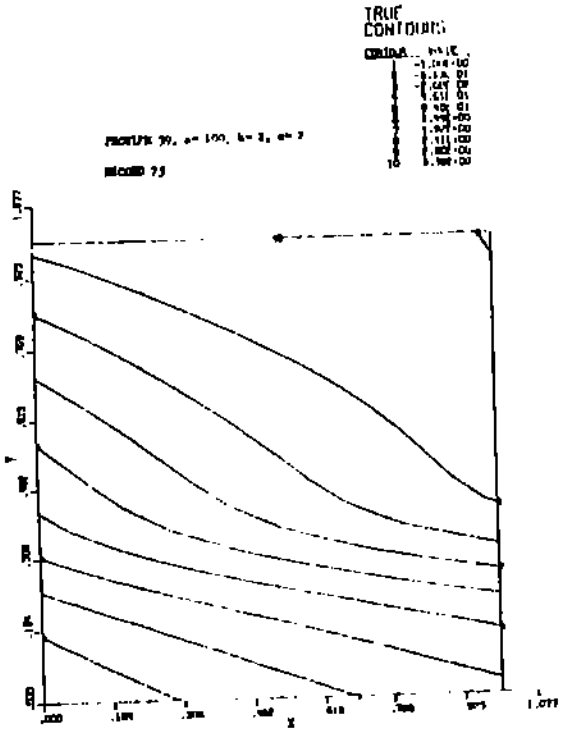

TRUE
CONTOURS

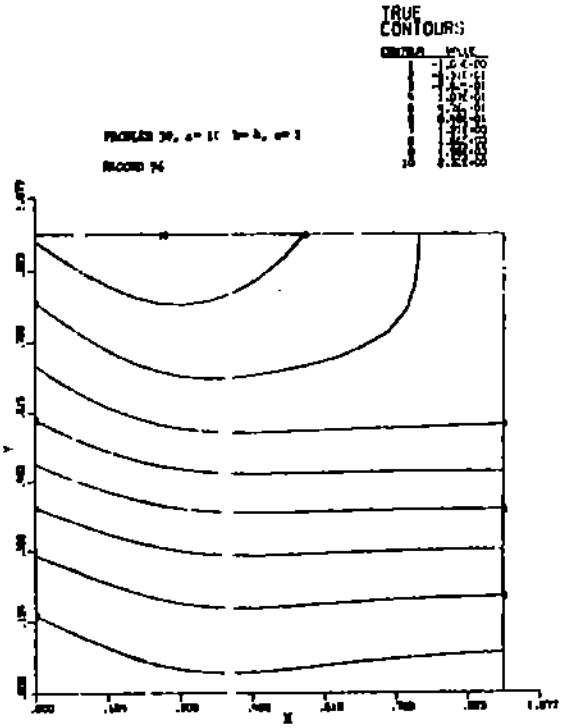

TRUE
CONTOURS

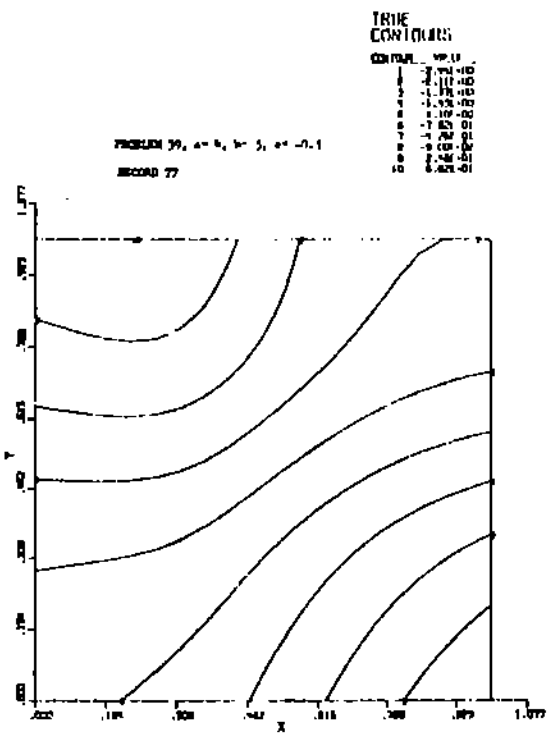

TRUE
CONTOURS

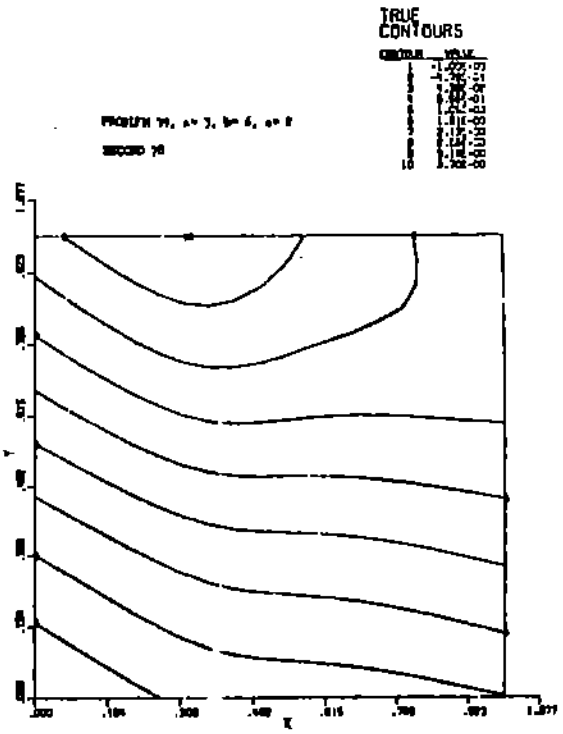

TRUE
CONTOURS

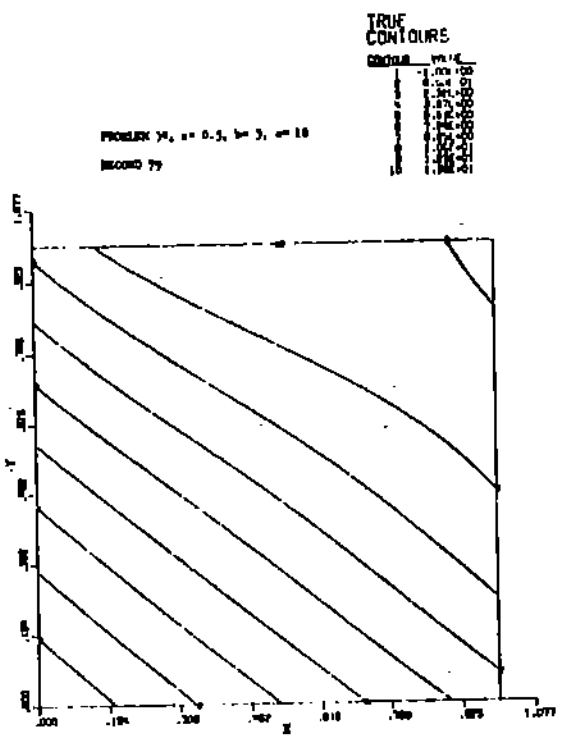

TRUE
CONTOURS


TRUE
CONTOURS


TRUE
CONTOURS


TRUE
CONTOURS


TRUE
CONTOURS


TRUE
CONTOURS
PROBLEM 74, a= 0.5, b= 3, n= 10
RECORD 79

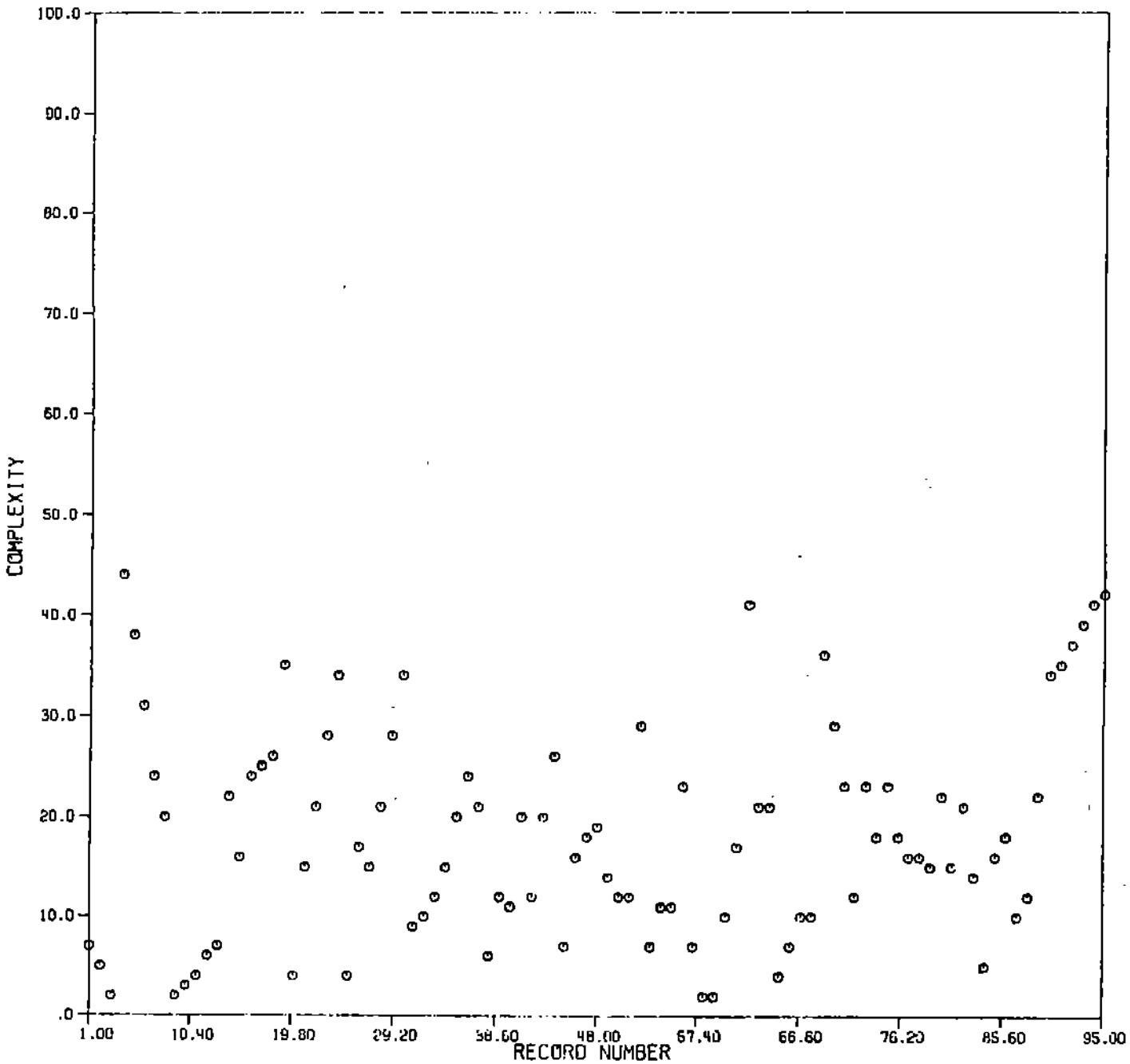
CROSS REFERENCE TABLE

Several records may use the same true solution. Only one contour plot is generated for each true solution, however. The following table should help you find plots that appear to be missing.

To get plot for record	Look at record
10	3
11	3
12	3
13	3
26	43
40	38
41	38
42	39
47	46
48	3
49	3
50	3
51	3
54	38
55	39
56	39
58	9
59	9

APPENDIX 2

ELLPACK PLUS FORTRAN PROBLEM FILES


100.0
90.0
80.0
70.0
60.0
50.0
40.0
30.0
20.0
10.0
.0
COMPLEXITY
1.00
10.40
19.80
29.20
38.60
48.00
57.40
66.80
76.20
85.60
95.00
RECORD NUMBER

EQNFIL file

```
RECORD 1
*         000.07    006.05    000.00    006.05
*         2000002002002
1         SELF-ADJOINT   $   TWO DIMENSIONS
1         EXP(X*Y) UXX$ + EXP(-X*Y) UYY$ - 1./(1.+X+Y) U$  =  F(X,Y)
2         HOMOGENEOUS   $   DIRICHLET
2         X=0.,U=0.
2         X=1.,U=0.
2         Y=0.,U=0.
2         Y=1.,U=0.
3     FUNCTION F(X,Y)
3     DATA PI/3.14159265358979/
3     PX = PI*X
3     PY = PI*Y
3     SPX = SIN(PX)
3     SPY = SIN(PY)
3     EXY = EXP(X*Y)
3     F = EXY*EXY*SPY*((2.*Y*Y-PI*PI)*SPX+3.*PI*Y*COS(PX))+
3    $    PI*SPX*(X*COS(PY)-PI*SPY)+EXY*SPX*SPY/(1.+X+Y)
3     RETURN
3     END
3     FUNCTION TRUE(X,Y)
3     DATA PI/3.14159265358979/
3     TRUE = .75*EXP(X*Y)*SIN(PI*X)*SIN(PI*Y)
3     RETURN
3     END
-------------------------------------------
RECORD 2
*         000.05    000.00    004.05    010.02
*         2000000000022
1         TWO DIMENSIONS
1         UXX$ + (1.+Y*Y) UYY$ - UX$ - (1.+Y*Y) UY
1             =  (-4.*X*X*X+18.*X*X-14.*X+2.)*ALOG(1.+Y*Y)
1.              - 2.*((X*X-X)**2)*(Y*Y+Y**3+Y-1.)/(1.+Y*Y)
2         HOMOGENEOUS   $   MIXED
2         X=0.,MIXED=(1.) U  (-1.) UX  =  0.
2         X=1.,MIXED=(1.) U  ( 1.) UY  =  0.
2         Y=0.,MIXED=(1.) U  ( 1.) UX  =  0.
2         Y=1.,MIXED=(1.) U  (-1.) UY  =  0.
3     FUNCTION TRUE(X,Y)
3     TRUE = .135*(EXP(X+Y)+(X*X-X)**2*ALOG(1.+Y*Y))
3     RETURN
3     END
-------------------------------------------
RECORD 3
*         000.02    004.00    000.00    006.00
*         2020221002002
1         POISSON   $   CONSTANT COEFFICIENTS   $   TWO DIMENSIONS
1         UXX$ + UYY$  =  6.*X*Y*EXP(X+Y)*(X*Y+X+Y-3.)
2         HOMOGENEOUS   $   DIRICHLET
2         X=0.,U=0.
2         X=1.,U=0.
2         Y=0.,U=0.
2         Y=1.,U=0.
3     FUNCTION TRUE(X,Y)
3     TRUE = 3.*EXP(X+Y)*X*Y*(1.-X)*(1.-Y)
3     RETURN
3     END
-------------------------------------------
RECORD 4
*         000.44    090.90    000.00    080.25
*         2000002002002
1         SELF - ADJOINT $ TWO DIMENSIONS
1         COEF1(X,Y)UXX $ + UYY = F(X,Y)
2         DIRICHLET
2         X =0. ,U = TRUE(0.,Y)
2         X =1. ,U = TRUE(1.,Y)
2         Y =0. ,U = TRUE(X,0.)
2         Y =1. ,U = TRUE(X,1.)
3      FUNCTION COEF1(X,Y)
3      COEF1 = 2.
3      IF( X.LT..4 ) COEF1 =1.
```

```
3       RETURN
3       END
3       FUNCTION F(X,Y)
3       FX1 = X + .3
3       FX2 = .7 + .5*(X-.4) + (X-.4)**2/(1+X*X)
3       D2FX1 = 0.
3       RX2P1 = 1./(1.+X**X)
3       D2FX2 = 2.*RX2P1-4.*X*(X-.4)*RX2P1**2-2.*(X-.4)**2*RX2P1**2
3     $       -4.*X*(X-.4)*RX2P1**2+4.*X*(X-.4)**2*RX2P1**3
3       F = COEF1(X,Y)*AMIN1(D2FX1,D2FX2)*(1.+(Y-1.)**2*EXP(-Y))
3     $   + AMIN1(FX1,FX2)*(7.-6.*Y+Y*Y)*EXP(-Y)
3       RETURN
3       END
3       FUNCTION TRUE(X,Y)
3       FX2 = .7 + .5*(X-.4) + (X-.4)**2/(1+X*X)
3       TRUE =AMIN1(X+.3,FX2)*(1.+(Y-1.)**2*EXP(-Y))
3       RETURN
3       END
-------------------------------------------------------
RECORD 5
*           000.38    090.60    000.00    070.40
EXPAND 1/1.5/
-------------------------------------------
RECORD 6
*           000.31    080.50    000.00    060.20
EXPAND 1/2.5/
-------------------------------------------
RECORD 7
*           000.24    070.30    000.00    050.15
EXPAND 1/3.5/
-------------------------------------------
RECORD 8
*           000.20    055.20    000.00    040.20
EXPAND 1/4.5/
-------------------------------------------
RECORD 9
*           000.02    002.00    000.00    006.00
EXPAND 2/0./
-------------------------------------------
RECORD 10
*           000.03    010.00    000.00    006.00
EXPAND 2/10./
-------------------------------------------
RECORD 11
*           000.04    015.00    000.00    006.05
EXPAND 2/20./
-------------------------------------------
RECORD 12
*           000.06    030.00    000.00    006.00
EXPAND 2/64./
-------------------------------------------
RECORD 13
*           000.07    040.00    000.00    006.00
EXPAND 2/100./
-------------------------------------------
RECORD 14
*           008.22    040.20    000.00    008.30
*           2000201002002
1           TWO DIMENSIONS
1           UXX$ + UYY$ + COEFU(X,Y) U  =  F(X,Y)
2           HOMOGENEOUS    $   DIRICHLET
2           X=0.,U=0.
2           X=1.,U=0.
2           Y=0.,U=0.
2           Y=1.,U=0.
3       FUNCTION TRUE(X,Y)
3       PI=3.1415926535897932
3       FOURP=4.*PI
3       FPX=FOURP*X
3       FPY=FOURP*Y
3       CX=COS(FPX)
3       CY=COS(FPY)
```

```
3      F1=-CX+5.4
3      F2=-CY+5.4
3      F3= (X-.5)*(X-.5)+(Y-.5)*(Y-.5)
3      F32=16.*F3*F3
3      F34=F32*F32
3      Z=1./(1.+F34)
3      F4=Z-.5
3      SPX=SIN(PI*X)
3      GOFY=Y*Y-Y
3      TRUE=F1*SPX*GOFY*F2*F4
3      RETURN
3      END
3      FUNCTION F(X,Y)
3      PI=3.141592653589793
3      FOURP=4.*PI
3      FPX=FOURP*X
3      FPY=FOURP*Y
3      SXTPSQ=FOURP*FOURP
3      SX=SIN(FPX)
3      SY=SIN(FPY)
3      CX=COS(FPX)
3      CY=COS(FPY)
3      F1=-CX+5.4
3      F2=-CY+5.4
3      DXF1=FOURP*SX
3      DYF2=FOURP*SY
3      DDXF1=SXTPSQ*CX
3      DDYF2=SXTPSQ*CY
3      F3= (X-.5)*(X-.5)+(Y-.5)*(Y-.5)
3      F32=16.*F3*F3
3      F33= F32*4.*F3
3      F34=F32*F32
3      Z=1./(1.+F34)
3      F4=Z-.5
3      DXF3= 2.*(X-.5)
3      DYF3= 2.*(Y-.5)
3      DDF3=2.
3      ZZ=Z*Z
3      W=F32*ZZ
3      W32=F33*ZZ
3      W6=W32*F33*Z
3      DF4=-16.*W32
3      DXF4=DF4*DXF3
3      DYF4=DF4*DYF3
3      A1=-192.*W
3      A2=-16.*W32*DDF3
3      A3=512.*W6
3      DXF3S=DXF3*DXF3
3      DYF3S=DYF3*DYF3
3      DDXF4=(A1+A3)*DXF3S+A2
3      DDYF4=(A1+A3)*DYF3S+A2
3      SPX=SIN(PI*X)
3      PICPX=PI*COS(PI*X)
3      GOFY=Y*Y-Y
3      DGOFY=2.*Y-1.
3      UXX=DDXF1*SPX*GOFY*F2*F4
3     *+DXF1*PICPX*GOFY*F2*F4
3     *+DXF1*SPX*GOFY*DXF4*F2
3     *+DXF1*PICPX*GOFY*F2*F4
3     *  -F1*PI*PI*SPX*GOFY*F2*F4
3     *  +F1*PICPX*GOFY*F2*DXF4
3     *+DXF1*SPX*GOFY*F2*DXF4
3     *+F1*PICPX*GOFY*F2*DXF4
3     *+F1*SPX*GOFY*F2*DDXF4
3      UYY=F1*SPX*DGOFY*DYF2*F4
3     *+F1*SPX*2.*F2*F4
3     *+F1*SPX*DGOFY*F2*DYF4
3     *+F1*SPX*DGOFY*DYF2*F4
3     *+F1*SPX*GOFY*DDYF2*F4
3     *+F1*SPX*GOFY*DYF2*DYF4
3     *+F1*SPX*DGOFY*F2*DYF4
3     *+F1*SPX*GOFY*DYF2*DYF4
```

```
3     *+F1*SPX*GOFY*F2*DDYF4
3      A=100.+COS(2.*PI*X)+SIN(3.*PI*Y)
3      A = - A
3      U=TRUE(X,Y)
3      F=UXX+UYY+A*U
3      RETURN
3       END
3      FUNCTION COEFU(X,Y)
3      PI=3.141592653589793
3      COEFU=100.+COS(2.*PI*X)+SIN(3.*PI*Y)
3      COEFU = -COEFU
3      RETURN
3      END
-------------------------------------------
RECORD 15
*          002.16     040.00     000.00     006.60
EXPAND 3/10./
-------------------------------------------
RECORD 16
*          002.24     040.45     000.00     006.65
EXPAND 3/20./
-------------------------------------------
RECORD 17
*          002.25     040.45     000.00     006.70
EXPAND 3/50./
-------------------------------------------
RECORD 18
*          002.26     040.45     000.00     006.80
EXPAND 3/100./
-------------------------------------------
RECORD 19
*          000.35     080.40     000.00     060.50
*          2020121002000
1          POISSON   $   CONSTANT COEFFICIENTS   $   TWO DIMENSIONS
1          UXX$ + UYY$  =  D2P(X)*P(Y)+P(X)*D2P(Y)
2          DIRICHLET
2          X=0.,U=TRUE(X,Y)
2          X=1.,U=TRUE(X,Y)
2          Y=0.,U=TRUE(X,Y)
2          Y=1.,U=TRUE(X,Y)
3      FUNCTION TRUE(X,Y)
3      TRUE = P(X)*P(Y)
3      RETURN
3      END
3       FUNCTION P(X)
3       A = 1.
3       B = 0.
3       E = .15
3       X1 = .5 - E
3       X2 = .5 + E
3       IF(X .LT. X1) GO TO 1
3       IF(X .GT. X2) GO TO 2
3       DPHI = B - A
3       DX = X2 - X1
3       P = A + DPHI*(X-X1)**3/(DX**3)-3.*DPHI*(X-X1)**3*(X-X2)
3     $ /DX**4 + 6.*DPHI*(X-X1)**3*(X-X2)**2/DX**5
3       RETURN
31      P = A
3       RETURN
32      P = B
3       RETURN
3       END
3       FUNCTION D2P(X)
3       A = 1.
3       B = 0.
3       E = .15
3       X1 = .5 - E
3       X2 = .5 + E
3       IF(X .LT. X1) GO TO 1
3       IF(X .GT. X2) GO TO 1
3       DPHI = B - A
3       DX = X2 - X1
```

```
3       C3 = DPHI/DX**3
3       C4 = -3.*DPHI/DX**4
3       C5 = 6.*DPHI/DX**5
3       D2P = 6.*C3*(X-X1)+6.*C4*(X-X1)*(X-X2)+
3     $       6.*C4*(X-X1)**2+6.*C5*(X-X1)*(X-X2)**2+
3     $       12.*C5*(X-X1)**2*(X-X2)
3     $   +       2.*C5*(X-X1)**3
3       RETURN
31      D2P = 0.
3       RETURN
3       END
-----------------------------------------
RECORD 20
*          000.04     005.10     000.00     005.05
EXPAND 4/10.,.5/
-----------------------------------------
RECORD 21
*          000.15     025.40     000.00     008.25
EXPAND 4/50.,.5/
-----------------------------------------
RECORD 22
*          000.21     040.60     000.00     010.30
EXPAND 4/100.,.5/
-----------------------------------------
RECORD 23
*          000.28     065.70     000.00     020.35
EXPAND 4/500.,.5/
-----------------------------------------
RECORD 24
*          000.34     080.75     000.00     035.40
EXPAND 4/1000.,.5/
-----------------------------------------
RECORD 25
*          000.04     005.10     000.00     005.05
EXPAND 4/10.,.117/
-----------------------------------------
RECORD 26
           000.17     080.00     000.00     005.30
EXPAND 18/30.,20./
-----------------------------------------
RECORD 27
*          000.15     025.40     000.00     008.25
EXPAND 4/50.,.117/
-----------------------------------------
RECORD 28
*          000.21     040.60     000.00     010.30
EXPAND 4/100.,.117/
-----------------------------------------
RECORD 29
*          000.28     065.70     000.00     020.35
EXPAND 4/500.,.117/
-----------------------------------------
RECORD 30
*          000.34     080.75     000.00     035.40
EXPAND 4/1000.,.117/
-----------------------------------------
RECORD 31
*          002.09     000.00     000.00     050.10
*          20202210020022 2
1          TWO DIMENSIONS $ POISSON $ CONSTANT COEFFICIENTS
1          UXX$ + UYY$ = 1.
2          HOMOGENEOUS $ DIRICHLET
2          X = 0.,U = 0.
2          X = 1.,U = 0.
2          Y = 0.,U = 0.
2          Y = 1.,U = 0.
3       FUNCTION TRUE(X,Y)
3         DIMENSION  XB(4), YB(4), AB(4), C(11)
3         DATA  N/16/, NZB/4/, NA/2/,
3      *     XB(1),YB(1), XB(2),YB(2), XB(3),YB(3), XB(4),YB(4), AB
3      *     / 0E0,0E0,     1E0,0E0,     1E0,1E0,     0E0,1E0,
3      *      -2.356E0,    -0.785E0,    +0.785E0,    +2.356E0/
```

```
3       DATA  C
3     *     / .1469642607553E+00, .2000000125850E+01, .1000000062928E+01,
3     *       -.2983663620768E-01,-.9430879889679E-03, .4398951975566E-05,
3     *       .2394115460351E-05, .3183099062144E+00,-.3183099062144E+00,
3     *       .3183099062144E+00,-.3183099062144E+00/
3       DATA  ERRMAX/6.2E-08/
3       DATA  PI/3.14159265358979323E0/, TWOPI/6.28318530717958646E0/
3       QABS (X, Y) = SQRT (X*X + Y*Y)
3
3       RCST = X - 1./2.
3       RSNT = Y - 1./2.
3       R = QABS (RCST, RSNT)
3       SUM = R*R / 4.
3
3       L = 1
3       SUM = SUM + C(L)
3       RKCSKT = 1
3       RKSNKT = 0
3       DO 1 K=1,N
3         TEMP   = RCST*RKCSKT - RSNT*RKSNKT
3         RKSNKT = RSNT*RKCSKT + RCST*RKSNKT
3         RKCSKT = TEMP
3         IF ((K.GT.2) .AND. (MOD(K,4).NE.0))  GO TO 1
3         L = L+1
3         SUM = SUM + C(L) * RKCSKT
3   1     CONTINUE
3
3       DO 2 I=1,NZB
3         ZR = X - XB(I)
3         ZI = Y - YB(I)
3         WR = QABS (ZR, ZI)
3         IF (WR.NE.0.)  WR = ALOG(WR)
3         WI = QARG (ZR, ZI)
3         IF (WI.GT.AB(I))  WI = WI - TWOPI
3         TR = ZR*WR - ZI*WI
3         TI = ZR*WI + ZI*WR
3         L = L+1
3         SUM = SUM + C(L) * (ZR*TI + ZI*TR)
3   2     CONTINUE
3       TRUE = SUM
3       RETURN
3       END
3       REAL FUNCTION  QARG (X, Y)
3       QARG = 0
3       IF ((X.NE.0.).OR.(Y.NE.0.))  QARG = ATAN2(Y,X)
3       RETURN
3       END
-----------------------------------------
RECORD 32
*         000.10    006.25    000.00    006.25
EXPAND 5/PI/
-----------------------------------------
RECORD 33
*         000.12    006.35    000.00    006.35
EXPAND 5/2.*PI/
-----------------------------------------
RECORD 34
*         000.15    006.40    000.00    006.40
EXPAND 5/3.*PI/
-----------------------------------------
RECORD 35
*         000.20    006.60    000.00    006.60
EXPAND 5/5.*PI/
-----------------------------------------
RECORD 36
*         000.24    006.75    000.00    006.75
EXPAND 5/10.*PI/
-----------------------------------------
RECORD 37
*         000.21    010.40    000.00    010.35
EXPAND 13/1.,8.,-1./
-----------------------------------------
```

```
RECORD 38
*         000.06    006.10    000.00    006.10
EXPAND 6/3.14159265358979,3.14159265358979/
-----------------------------------------
RECORD 39
*         000.12    006.30    000.00    006.30
EXPAND 6/3.14159265358979,10./
-----------------------------------------
RECORD 40
*         000.11    006.30    000.00    006.30
EXPAND 6/10.,3.14159265358979/
-----------------------------------------
RECORD 41
*         000.20    006.60    000.00    006.60
EXPAND 6/20.,3.14159265358979/
-----------------------------------------
RECORD 42
*         000.12    006.30    000.00    006.30
EXPAND 6/10.,10./
-----------------------------------------
RECORD 43
*         000.20    006.60    000.00    006.60
EXPAND 6/10.,20./
-----------------------------------------
RECORD 44
*         008.26    098.60    000.00    006.05
*         200000000200222
1         TWO DIMENSIONS
1         UXX$ + 1./X**2 UYY$ + 2./X UX$ + G(X,Y) UY$ + U=F(X,Y)
2         DIRICHLET
2         X = 0.    , U = TRUE(X,Y)
2         X = 1.    , U = TRUE(X,Y)
2         Y = 0.    , U = TRUE(X,Y)
2         Y = 1.    , U = TRUE(X,Y)
3     FUNCTION TRUE(X,Y)
3     TRUE = EXP(X+Y)
3     RETURN
3     END
3     FUNCTION G(X,Y)
3     G = 1./(X*ATAN(Y))
3     RETURN
3     END
3     FUNCTION F(X,Y)
3     F = 1. + 1./X**2 + 2./X + G(X,Y) + 1.
3     F = F * EXP(X+Y)
3     RETURN
3     END
-----------------------------------------
RECORD 45
*         008.07    010.20    000.00    002.10
*         200000002200022
1         TWO DIMENSIONS
1         A(X,Y)UXX$ + B(X,Y)UXY$ + C(X,Y)UYY = F(X,Y)
2         DIRICHLET
2         X = 0. , U=TRUE(X,Y)
2         X = 1. , U=TRUE(X,Y)
2         Y = 0. , U=TRUE(X,Y)
2         Y = 1. , U=TRUE(X,Y)
3     FUNCTION A(X,Y)
3     A = 1. + TRUEX(X,Y)**2
3     RETURN
3     END
3     FUNCTION B(X,Y)
3     B = -2.*TRUEX(X,Y)*TRUEY(X,Y)
3     RETURN
3     END
3     FUNCTION C(X,Y)
3     C = 1. + TRUEY(X,Y)**2
3     RETURN
3     END
3     FUNCTION TRUEX(X,Y)
3     TRUEX = 2.*(X-3.*Y)
```

```
3       RETURN
3       END
3       FUNCTION TRUEY(X,Y)
3       TRUEY = -6.*(X-3.*Y)
3       RETURN
3       END
3       FUNCTION F(X,Y)
3       TUXX = 2.
3       TUXY = -6.
3       TUYY = 18.
3       F = A(X,Y)*TUXX+B(X,Y)*TUXY+C(X,Y)*TUYY
3       RETURN
3       END
3       FUNCTION TRUE(X,Y)
3       TRUE = (X-3.*Y)**2
3       RETURN
3       END
-----------------------------------------
RECORD 46
*           008.16    070.10    000.00    006.05
EXPAND 7/0.15/
-----------------------------------------
RECORD 47
*           008.18    080.10    000.00    006.05
EXPAND 7/.25/
-----------------------------------------
RECORD 48
*           000.19    065.40    000.00    006.05
EXPAND 8/1.5/
-----------------------------------------
RECORD 49
*           000.14    055.20    000.00    006.05
EXPAND 8/2.5/
-----------------------------------------
RECORD 50
*           000.12    045.15    000.00    006.05
EXPAND 8/3.5/
-----------------------------------------
RECORD 51
*           000.12    035.20    000.00    006.05
EXPAND 8/4.5/
-----------------------------------------
RECORD 52
*           008.29    090.85    000.00    004.05
*           2000002002002
1           SELF-ADJOINT $ TWO DIMENSIONS
1           P(X,Y)UXX$ + P(X,Y)UYY$ + DXP(X,Y)UX $+DXP(X,Y)UY=F(X,Y)
2           DIRICHLET
2           X = 0. , U=TRUE(0.,Y)
2           X = 1.,  U =TRUE(1.,Y)
2           Y = 0. , U =TRUE(X,0.)
2           Y = 1. , U =TRUE(X,1.)
3       FUNCTION P(X,Y)
3       COMMON /PARS/ C1,C2
3       DATA C1,C2/.01706E+3,.00362E+3/
3       F1 = (X*X-1.)*(Y*Y-1.)
3       DXF1 = 2.*X*(Y**2-1.)
3       DYF1 = (X*X-1.)*2.*Y
3       UX = C1*DXF1+C2*DXF1*(X*X+Y*Y)+C2*F1*2.*X
3       UY = C1*DYF1+C2*DYF1*(X*X+Y*Y)+C2*F1*2.*X
3       A = (UX*UX + UY*UY )**.5
3       P = 7.996E+3
3       IF( A .GT. .0025 ) P = 19.5/A+236.
3       RETURN
3       END
3       FUNCTION DXP(X,Y)
3       COMMON /PARS/ C1,C2
3       F1 = (X*X-1.)*(Y*Y-1.)
3       DXF1 = 2.*X*(Y**2-1.)
3       DYF1 = (X*X-1.)*2.*Y
3       UX = C1*DXF1+C2*DXF1*(X*X+Y*Y)+C2*F1*2.*X
3       UY = C1*DYF1+C2*DYF1*(X*X+Y*Y)+C2*F1*2.*X
```

```
3       A = (UX*UX + UY*UY )**.5
3       DXP = 0.
3       IF( A .GT. .0025 ) DXP =-19.5/A**2
3       RETURN
3       END
3       FUNCTION TRUE(X,Y)
3       COMMON /PARS/ C1,C2
3       F1 = (X*X-1.)*(Y*Y-1.)
3       TRUE = C1*F1+C2*F1*(X*X+Y*Y)
3       RETURN
3       END
3       FUNCTION F(X,Y)
        COMMON /PARS/ C1,C2
3       F1 = (X*X-1.)*(Y*Y-1.)
3       DXF1 = 2.*X*(Y**2-1.)
3       DYF1 = (X*X-1.)*2.*Y
3       UX = C1*DXF1+C2*DXF1*(X*X+Y*Y)+C2*F1*2.*X
3       UY = C1*DYF1+C2*DYF1*(X*X+Y*Y)+C2*F1*2.*X
3       DXXF1 = 2.*(Y*Y-1.)
3       DYYF1 = (X*X-1.)*2.
3       UXX = C1*DXXF1+C2*DXXF1*(X*X+Y*Y)+C2*DXF1*2.*X+C2*DXF1*2.*X
3     $       +C2*DXF1*2.
3       UYY = C1*DYYF1+C2*DYYF1*(X*X+Y*Y)+C2*DYF1*2.*Y
3     $     +C2*DYF1*2.*Y+C2*DYF1*2.
3       PU = P(X,Y)
3       DPU = DXP(X,Y)
3       F = PU*(UXX+UYY)+DPU*(UX+UY)
3       RETURN
3       END
-----------------------------------------
RECORD 53
*         009.07    010.05    000.00    010.15
*         2000002002002
1         SELF - ADJOINT $ TWO DIMENSIONS
1         UXX $ + UYY $ + 3./(5.-Y)UY = F(X,Y)
2         DIRICHLET
2         X = -.5 , U = 0.
2         X =  .5 , U = 0.
2         Y = -1. , U = 0.
2         Y =  1. , U = 0.
3     FUNCTION TRUE(X,Y)
3     DATA A,B /.0010185,.0004838/
3     TRUE = (1.-Y*Y)*(1.-4.*X*X)*(5.-Y)**3*(A+B*Y)
3     RETURN
3     END
3     FUNCTION F(X,Y)
3     COMMON /PARS/ A,B
3     U = (1.-Y**2)*(1.-4.*X**2)*(5.-Y)**3*(A+B*Y)
3     UX = -8.*X*(1-Y**2)*(5.-Y)**3*(A+B*Y)
3     UXX = -8.*(1.-Y**2)*(5.-Y)**3*(A+B*Y)
3     UY = (1.-4.*X**2)*(B-2.*A*Y-3.*B*Y**2)*(5.-Y)**3
3   $     -(1.-4.*X**2)*(A+B*Y-A*Y**2-B*Y**3)*3.*(5.-Y)**2
3     UYY = (1.-4.*X**2)*(-2.*A-6.*B*Y)*(5.-Y)**3
3   $      -(1.-4.*X**2)*(B-2.*A*Y-3.*B*Y**2)*3.*(5.-Y)**2
------------ ------------------------
RECORD 54
*         000.11    006.30    000.00    006.30
EXPAND 18/10.,3.14159265358979/
-----------------------------------------
RECORD 55
*         000.11    006.30    000.00    006.30
EXPAND 18/10.,10./
-----------------------------------------
RECORD 56
*         000.23    006.75    000.00    006.75
EXPAND 18/30.,10./
-----------------------------------------
RECORD 57
*         008.07    010.20    000.00    006.10
*         200000002200022
1         TWO DIMENSIONS
1         A(X,Y)UXX$ + B(X,Y)UXY$ + C(X,Y)UYY = F(X,Y)
```

```
2         DIRICHLET
2         X = 0. , U=TRUE(X,Y)
2         X = 1. , U=TRUE(X,Y)
2         Y = 0. , U=TRUE(X,Y)
2         Y = 1. , U=TRUE(X,Y)
3     FUNCTION A(X,Y)
3     A = 1. + TRUEX(X,Y)**2
3     RETURN
3     END
3     FUNCTION B(X,Y)
3     B = -2.*TRUEX(X,Y)*TRUEY(X,Y)
3     RETURN
3     END
3     FUNCTION C(X,Y)
3     C = 1. + TRUEY(X,Y)**2
3     RETURN
3     END
3     FUNCTION TRUEX(X,Y)
3     TRUEX = EXP(X+Y)
3     RETURN
3     END
3     FUNCTION TRUEY(X,Y)
3     TRUEY = EXP(X+Y)
3     RETURN
3     END
3     FUNCTION F(X,Y)
3     TUXX = EXP(X+Y)
3     TUXY = TUXX
3     TUYY = TUXY
3     F = A(X,Y)*TUXX+B(X,Y)*TUXY+C(X,Y)*TUYY
3     RETURN
3     END
3     FUNCTION TRUE(X,Y)
3     TRUE = EXP(X+Y)
3     RETURN
3     END
-----------------------------------
RECORD 58
*         000.02    002.00    000.00    006.00
EXPAND 2/5./
-----------------------------------
RECORD 59
*         000.02    002.00    000.00    006.00
EXPAND 2/8./
-----------------------------------
RECORD 60
*         000.10    002.20    000.00    010.15
EXPAND 10/1..2./
-----------------------------------
RECORD 61
*         000.17    010.50    000.00    020.40
EXPAND 10/5..3./
-----------------------------------
RECORD 62
*         000.41    040.80    000.00    080.70
EXPAND 10/8..5./
-----------------------------------
RECORD 63
*         007.21    040.60    000.00    010.30
EXPAND 11/0./
-----------------------------------
RECORD 64
*         007.21    040.60    000.00    010.30
EXPAND 11/10./
-----------------------------------
RECORD 65
*         009.04    002.00    004.08    006.05
*         2020221000022
1         POISSON   $   CONSTANT COEFFICIENTS   $   TWO DIMENSIONS
1         UXX$ + UYY$ = -1.
2         MIXED
2         X=-1. , MIXED = (-1.) UX (1.) U = -DU1(-1.,Y)+TRUE(-1.,Y)
```

```
2         X= 1. , MIXED = ( 1.) UX (1.) U =  DU1( 1.,Y)+TRUE( 1.,Y)
2         Y=-1. , MIXED = (-1.) UY (1.) U = -DU1(-1.,X)+TRUE(X,-1.)
2         Y= 1. , MIXED = ( 1.) UY (1.) U =  DU1( 1.,X)+TRUE(X, 1.)
3      FUNCTION TRUE(X,Y)
3      COMMON /PARS/ A0,A2,A4,A6
3      DATA A0,A2,A4,A6 /.821564,-.0144,.0000493,-.00000064/
3      X2 = X*X
3      X4 = X2*X2
3      X6 = X4*X2
3      X8 = X6*X2
3      X10 = X8*X2
3      X12 = X10*X2
3      Y2 = Y*Y
3      Y4 = Y2*Y2
3      Y6 = Y4*Y2
3      Y8 = Y6*Y2
3      Y10 = Y8*Y2
3      Y12 = Y10*Y2
3      TRUE = -.25*(X2+Y2) + A0 + A2*(X4-6.*X2*Y2+Y4)
3     $        + A4*(X8-28.*X6*Y2+70.*X4*Y4-28.*X2*Y6+Y8)
3     $        + A6*(X12-66.*X10*Y2+495.*X8*Y4-924.*X6*Y6+
3     $              Y12-66.*X2*Y10+495.*X4*Y8              )
3      RETURN
3      END
3      FUNCTION DU1(X,Y)
3      COMMON /PARS/ A0,A2,A4,A6
3      X2 = X*X
3      X3 = X*X2
3      X5 = X3*X2
3      X7 = X5*X2
3      X9 = X7*X2
3      X11 = X9*X2
3      Y2 = Y*Y
3      Y4 = Y2*Y2
3      Y6 = Y4*Y2
3      Y8 = Y6*Y2
3      Y10 = Y8*Y2
3      DU1 = -.5*X + A2*(4.*X3-12.*X*Y2) +
3     $       A4*(8.*X7-168.*X5*Y2+280.*X3*Y4-56.*X*Y6) +
3     $       A6*(12.*X11-660.*X9*Y2+3960.*X7*Y4-5544.*X5*Y6+
3     $           1980.*X3*Y8-132.*X*Y10)
3      RETURN
3      END
-----------------------------------------
RECORD 66
*         008.07    002.01    000.01    006.05
*         2020221002000
1         POISSON   $   CONSTANT COEFFICIENTS   $   TWO DIMENSIONS
1          UXX$ + UYY$ = -1.
2         DIRICHLET
          X=-1. , U=TRUE(-1.,Y)   $   X=1. , U=TRUE(1.,Y)
2         Y=-1. , U=TRUE(X,-1.)   $   Y=1. , U=TRUE(X,1.)
3      FUNCTION TRUE(X,Y)
3      X2 = X*X
3      X4 = X2*X2
3      X6 = X4*X2
3      X8 = X6*X2
3      Y2 = Y*Y
3      Y4 = Y2*Y2
3      Y6 = Y4*Y2
3      Y8 = Y6*Y2
3      TRUE = 0.295776 - .25*(X2+Y2) +
3     $        (-14476.*(X4-6.*X2*Y2+Y4) +
3     $          429.*(X8-28.*X6*Y2+70.*X4*Y4-28.*X2*Y6+Y8))/319424.
3      RETURN
3      END
-----------------------------------------
RECORD 67
*         008.10    006.20    000.10    006.25
*         2022221202000
1         TWO DIMENSIONS   $   POISSON   $   CONSTANT COEFFICIENTS
1         UXX$ + UYY$ = F(X,Y)
```

```
2         DIRICHLET
2         X=0. , U=TRUE(0.,Y)   $   X=1. , U=TRUE(1.,Y)
2         Y=-1. , U=TRUE(X,-1.)   $   Y=1. , U=TRUE(X,1.)
3     FUNCTION TRUE(X,Y)
3     S133 = SQRT(133.)
3     K1 = SQRT(14.+S133)
3     K2 = SQRT(14.-S133)
3     A = (-7.+S133)/(2.*S133)
3     B = (-7.-S133)*A/16.
3     EK1X = EXP(K1*X)
3     EK2X = EXP(K2*X)
3     EDIFF = EK1X - EK2X
3     F1 = A*EDIFF + EK2X
3     F2 = B*EDIFF
3     Y21 = 1.-Y*Y
3     TRUE = Y21*(F1 + Y21*F2)
3     RETURN
3     END
3     FUNCTION F(X,Y)
3     S133 = SQRT(133.)
3     K1 = SQRT(14.+S133)
3     K2 = SQRT(14.-S133)
3     A = (-7.+S133)/(2.*S133)
3     B = (-7.-S133)*A/16.
3     EK1X = EXP(K1*X)
3     EK2X = EXP(K2*X)
3     EDIFF = EK1X-EK2X
3     F1 = A*EDIFF + EK2X
3     F2 = B*EDIFF
3     Y21 = 1.-Y*Y
3     DEK1X = K1*K1*EK1X
3     DEK2X = K2*K2*EK2X
3     DEDIFF = DEK1X - DEK2X
3     DDF1 = A*DEDIFF + DEK2X
3     DDF2 = B*DEDIFF
3     UXX = Y21*(DDF1 + Y21*DDF2)
3     UYY = -2.*(F1 + 2.*(1.-3.*Y*Y)*F2)
3     F = UXX + UYY
3     RETURN
3     END
-----------------------------------------
RECORD 68
*         009.10    002.00    000.10    006.50
*         2022221202000
1         LAPLACE   $   POISSON   $   CONSTANT COEFFICIENTS
1         TWO DIMENSIONS
1         UXX$ + UYY$ = 0.
2         DIRICHLET
2         X=-1. , U=TRUE(-1.,Y)   $   X=1. , U=TRUE(1.,Y)
2         Y=-1. , U=TRUE(X,-1.)   $   Y=1. , U=TRUE(X,1.)
3     FUNCTION TRUE(X,Y)
3     X2 = X*X
3     X4 = X2*X2
3     X6 = X4*X2
3     X8 = X6*X2
3     Y2 = Y*Y
3     Y4 = Y2*Y2
3     Y6 = Y4*Y2
3     Y8 = Y6*Y2
3     U1 = X4 - 6.*X2*Y2 + Y4
3     U2 = X8 - 28.*X6*Y2 + 70.*X4*Y4 - 28.*X2*Y6 + Y8
3     U3 = U2 + 3.*U1
3     TRUE = 1.1786 - .1981*U1 + .008*U3
3     RETURN
3     END
-----------------------------------------
RECORD 69
*         000.36    090.60    000.10    070.35
EXPAND 12/3./
-----------------------------------------
RECORD 70
*         000.29    080.50    000.00    060.15
```

```
EXPAND 12/5./
-------------------------------------
RECORD 71
*         000.23    070.30    000.00    050.10
EXPAND 12/7./
-------------------------------------
RECORD 72
*         000.12    006.10    000.00    006.15
EXPAND 13/1.,2.,.5/
-------------------------------------
RECORD 73
          000.23    030.30    000.00    030.25
EXPAND 13/23.,2.,1./
-------------------------------------
RECORD 74
*         000.18    010.30    000.00    015.25
EXPAND 13/10.,11.,0./
-------------------------------------
RECORD 75
*         000.23    030.30    000.00    030.25
EXPAND 13/100.,2.,2./
-------------------------------------
RECORD 76
*         000.18    010.30    000.00    015.25
EXPAND 13/10.,4.,1./
-------------------------------------
RECORD 77
*         000.16    006.25    000.00    010.20
EXPAND 13/4.,5.,-0.5/
-------------------------------------
RECORD 78
*         000.16    006.25    000.00    006.25
EXPAND 13/3.,6.,2./
-------------------------------------
RECORD 79
*         000.15    006.20    000.00    006.20
EXPAND 13/.5,3.,10./
-------------------------------------
RECORD 80
*         000.22    095.00    000.00    070.00
EXPAND 14/.2,.1,1.5/
-------------------------------------
RECORD 81
*         000.15    050.00    000.00    050.00
EXPAND 14/1.,.1,2.5/
-------------------------------------
RECORD 82
*         000.21    090.00    000.00    070.00
EXPAND 14/.2,.04,1.5/
-------------------------------------
RECORD 83
*         000.14    040.00    000.00    050.00
EXPAND 14/.2,.04,2.5/
-------------------------------------
RECORD 84
*         000.05    005.00    010.00    005.00
EXPAND 15/0./
-------------------------------------
RECORD 85
*         000.16    060.00    010.00    050.00
EXPAND 15/.1/
-------------------------------------
RECORD 86
*         000.18    070.00    010.00    070.00
EXPAND 15/1./
-------------------------------------
RECORD 87
*         000.10    010.10    000.00    010.10
EXPAND 18/1.,2./
-------------------------------------
RECORD 88
*         000.12    010.20    000.00    020.15
```

```
EXPAND 18/5.,3./
-----------------------------------------
RECORD 89
*          009.22     010.70     000.00     070.40
EXPAND 18/8.,5./
-----------------------------------------
RECORD 90
*          009.34     060.70     020.00     070.60
EXPAND 17/1,387.75,50./
-----------------------------------------
RECORD 91
*          009.35     070.80     020.00     070.60
EXPAND 17/1,554.5,.544/
-----------------------------------------
RECORD 92
*          009.37     080.80     020.00     070.60
EXPAND 17/2,387.75,50./
-----------------------------------------
RECORD 93
*          009.39     080.80     020.00     070.60
EXPAND 17/2,554.5,.544/
-----------------------------------------
RECORD 94
*          009.41     085.80     020.00     070.60
EXPAND 17/3,387.75,50./
-----------------------------------------
RECORD 95
*          009.42     085.85     020.00     070.60
EXPAND 17/3,554.5,.544/
-----------------------------------------
```

MACFIL file

```
MACRO 1
#          2020221002002
1          POISSON   $   CONSTANT COEFFICIENTS   $   TWO DIMENSIONS
1          UXX$ + UYY$  =  F(X,Y)
2          HOMOGENEOUS   $   DIRICHLET
2          X=0.,U=0.
2          X=1.,U=0.
2          Y=0.,U=0.
2          Y=1.,U=0.
3      FUNCTION TRUE(X,Y)
3      COMMON /PARAM/ C,CAA1,AM2,INITL
3      DATA INITL/1/
3      IF (INITL .EQ. 0)  GO TO 10
3         C = 1./(^A**(^A/(1.-^A))
3     $           -^A**(1./(1.-^A)))**2
3         CAA1 = C*^A*(^A-1.)
3         AM2 = ^A-2.
3         INITL = 0
3   10 CONTINUE
3      TRUE = C*(X**^A-X)*(Y**^A-Y)
3      RETURN
3      END
3      FUNCTION F(X,Y)
3      COMMON /PARAM/ C,CAA1,AM2,INITL
3      IF (INITL .EQ. 0)  GO TO 10
3         C = 1./(^A**(^A/(1.-^A))
3     $           -^A**(1./(1.-^A)))**2
3         CAA1 = C*^A*(^A-1.)
3         AM2  = ^A-2.
3         INITL = 0
3   10 CONTINUE
3      IF( X.EQ.0. .OR. Y.EQ.0. ) GO TO 20
3      F = CAA1*(X**AM2*(Y**^A-Y) +
3     $           Y**AM2*(X**^A-X)    )
3      RETURN
3  20  F = 0.
       RETURN
3      END
-------------------------------------
MACRO 2
#          2000221002002
1          CONSTANT COEFFICIENTS   $   TWO DIMENSIONS
1          4. UXX$ + UYY$ -^A U  =  F(X,Y)
2          HOMOGENEOUS   $   DIRICHLET
2          X=0.,U=0.
2          X=1.,U=0.
2          Y=0.,U=0.
2          Y=1.,U=0.
3      FUNCTION TRUE(X,Y)
3      TRUE = 2.*X*(X-1.)*(COS(6.2831853071795S*Y)-1.)
3      RETURN
3      END
3      FUNCTION F(X,Y)
3      DATA TWOPI/6.2831853071795S/
3      CTPY = COS(TWOPI*Y)
3      F = 2.*X*(X-1.)*(^A*(1.-CTPY)-TWOPI*TWOPI*CTPY)
3     $          +16.*(CTPY-1.)
3      RETURN
3      END
-------------------------------------
MACRO 3
#          2000221002000
1          CONSTANT COEFFICIENTS   $   TWO DIMENSIONS
1          UXX$ + UYY$ - 100. U  =  .5*(^A**2-100.)*COSH(^A*Y)/COSH(^A)
2          DIRICHLET
2          X=0.,U=TRUE(X,Y)
2          X=1.,U=TRUE(X,Y)
2          Y=0.,U=TRUE(X,Y)
2          Y=1.,U=TRUE(X,Y)
3      FUNCTION TRUE(X,Y)
3      TRUE = .5*(COSH(10.*X)/COSH(10.)+COSH(^A*Y)/COSH(^A))
3      RETURN
```

```
3     END
3     FUNCTION COSH(Z)
3     COSH = .5*(EXP(Z)+EXP(-Z))
3     RETURN
3     END
-------------------------------------
MACRO 4
*          2020221002000
1          POISSON   $   CONSTANT COEFFICIENTS   $   TWO DIMENSIONS
1          UXX$ + UYY$  =  F(X,Y)
2          DIRICHLET
2          X=0.,U=TRUE(X,Y)
2          X=1.,U=TRUE(X,Y)
2          Y=0.,U=TRUE(X,Y)
2          Y=1.,U=TRUE(X,Y)
3     FUNCTION TRUE(X,Y)
3     TRUE = EXP(-^A*((X-.5)**2+(Y-^B)**2))
3     RETURN
3     END
3     FUNCTION F(X,Y)
3     TEMP = ^A*((X-.5)**2+(Y-^B)**2)
3     F = 4.*^A*(TEMP-1.)*EXP(-TEMP)
3     RETURN
3     END
-------------------------------------
MACRO 5
*          2020121002000
1          POISSON   $   CONSTANT COEFFICIENTS   $   TWO DIMENSIONS
1          UXX$ + UYY$  =  F(X,Y)
2          DIRICHLET
2          X=0.,U=TRUE(X,Y)
2          X=1.,U=TRUE(X,Y)
2          Y=0.,U=TRUE(X,Y)
2          Y=1.,U=TRUE(X,Y)
3     FUNCTION TRUE(X,Y)
3     DATA PI/3.14159265358979/
3     TEMP = X-Y+2.
3     TEMP4 = TEMP**4
3     TRUE = SIN(^A*TEMP4*TEMP/(1.+TEMP4))
3     RETURN
3     END
3     FUNCTION F(X,Y)
3     DATA PI/3.14159265358979/
3     W = X-Y+2.
3     R = W**5
3     S = 1.+W**4
3     V = R/S
3     AV = ^A*V
3     RX = 5.*W**4
3     RXX = 20.*W**3
3     SX = 4.*W**3
3     SXX = 12 *W**2
3     S2 = S*S
3     S3 = S2*S
3     VX = (S*RX-R*SX)/S2
3     VXX = (S2*RXX-R*S*SXX-2.*S*SX*RX+2.*R*SX**2)/S3
3     CAV = COS(AV)
3     SAV = SIN(AV)
3     UXX = ^A*(CAV*VXX-^A*SAV*VX*VX)
3     F = 2.0*UXX
3     RETURN
3     END
-------------------------------------
MACRO 6
*          2000000002000
1          TWO DIMENSIONS
1          UXX$ + UYY$ + (1.+SIN(^A*X)) UX$
1                  - COS(^A*Y) U  =  F(X,Y)
2          DIRICHLET
2          X=0.,U=TRUE(X,Y)
2          X=1.,U=TRUE(X,Y)
2          Y=0.,U=TRUE(X,Y)
```

```
2          Y=1.,U=TRUE(X,Y)
3       FUNCTION TRUE(X,Y)
3       TRUE = COS(^B*Y)+SIN(^B*(X-Y))
3       RETURN
3       END
3       FUNCTION F(X,Y)
3       BXMY = ^B*(X-Y)
3       UX = ^B*COS(BXMY)
3       B2 = ^B*^B
3       UXX = -B2*SIN(BXMY)
3       U = COS(^B*Y)+SIN(BXMY)
3       UYY = -B2*U
3       F = UXX+UYY+(1.+SIN(^A*X))*UX
3      $            -COS(^A*Y)*U
3       RETURN
3       END
```

```
MACRO 7
*          200020100022222
1          TWO DIMENSIONS
1          W(X,Y)UXX$+W(X,Y)UYY$+WX(X,Y)UX$+WY(X,Y)UY=F(X,Y)
2          MIXED
2          X = .5 , MIXED = (1.)UX = 0.
2          X = 1. , MIXED = (1.)U = 0.
2          Y = .5 , MIXED = (1.)UY = 0.
2          Y = 1. , MIXED = (1.)U = 0.
3       FUNCTION W(X,Y)
3       PI = ACOS(-1.)
3       W = ((PI*COS(PI*X)*SIN(PI*Y))**2 +
3      $       (PI*SIN(PI*X)*COS(PI*Y))**2)**^A
3       RETURN
3       END
3       FUNCTION TRUE(X,Y)
3       PI = ACOS(-1.)
3       TRUE = SIN(PI*X)*SIN(PI*Y)
3       RETURN
3       END
3       FUNCTION WX(X,Y)
3       PI = ACOS(-1.)
3       WX = ^A*W(X,Y)**(1.-1./^A)*SIN(2.*PI*X)*COS(PI*2.*Y)
3       RETURN
3       END
3       FUNCTION WY(X,Y)
3       WY = ^A*W(X,Y)**(1.-1./^A)*SIN(2.*PI*Y)*COS(2.*PI*X)
3       RETURN
3       END
3       FUNCTION F(X,Y)
3       PI = ACOS(-1.)
3       PI2 = PI * PI
3       SINPIX = SIN(PI*X)
3       SINPIY = SIN(PI*Y)
3       COSPIX = COS(PI*X)
3       COSPIY = COS(PI*Y)
3       TU = SINPIX*SINPIY
3       TUX = PI*COSPIX*SINPIY
3       TUXX = -PI2*TU
3       TUY = PI*SINPIX*COSPIY
3       TUYY = -PI2*TU
3       F = W(X,Y)*(TUXX + TUYY) + WX(X,Y)*TUX + WY(X,Y)*TUY
3       RETURN
3       END
```

```
MACRO 8
*          200000000200222
1          TWO DIMENSIONS
1          -X**^A UXX$ -Y**^A UYY$ +^A*X**(^A-1.) UX $
1.         +^A*Y**(^A-1.)UY$ + (X*Y)**^AU = F(X,Y)
2          HOMOGENEOUS $ DIRICHLET
2          X = 0. , U = 0.
2          X = 1. , U = 0.
2          Y = 0. , U = 0.
2          Y = 1. , U = 0.
```

```
3      FUNCTION TRUE(X,Y)
3      TRUE = 3.*EXP(X)*EXP(Y)*(X-X*X)*(Y-Y*Y)
3      RETURN
3      END
3      FUNCTION F(X,Y)
3      EXPX = EXP(X)
3      EXPY = EXP(Y)
3      EXEY3 = 3.*EXPX*EXPY
3      YMYY = Y - Y*Y
3      XMXX = X - X*X
3      TU = EXEY3*XMXX*YMYY
3      TUX = EXEY3*(1. - X -X*X)*YMYY
3      TUXX = EXEY3*(-3*X- X*X)*YMYY
3      TUY = EXEY3*(1. - Y - Y*Y)*XMXX
3      TUYY = EXEY3*(-3*Y-Y*Y)*XMXX
3      F = -X** ^A *TUXX - Y** ^A * TUYY + ^A * X**( ^A-1.)*TUX +
3     $     ^A * Y**( ^A-1.) * TUY + (X*Y)** ^A *TU
3      RETURN
3      END
```

```
MACRO 9
*          2000200002000
1          TWO DIMENSIONS
1          UXX $ + (1.+X*X)UYY $ -YUX = F(X,Y)
2          MIXED
2          X =0. ,MIXED = (^A)U - (^B)UX =0.
2          X =1. ,MIXED = (^A)U - (^B)UX = 0.
2          Y =0. ,U = TRUE(X,0.)
2          Y =1. ,U = TRUE(X,1.)
3      FUNCTION TRUE(X,Y)
3      TRUE = EXP(2.*(X+Y)/(2.+X-Y)-2.)+ALOG10((X+1.)/(Y+1.))
3      RETURN
3      END
3      FUNCTION F(X,Y)
3      P= 2.*(X+Y)/(2.+X-Y) -2.
3      DXP = 4.*(1.-Y)/(2.+X-Y)**2
3      DXXP = -2.*(4.-4.*Y)/(2.+X-Y)**3
3      DYP = 4.*(1.+X)/(2.+X-Y)**2
3      U = EXP(P) +ALOG10((X+1.)/(Y+1.))
3      UX = DXP*EXP(P) +1./(X+1.)
3      UXX = DXXP*EXP(P)+DXP**2*EXP(P)-1./(X+1.)**2
3      UYY = DYYP*EXP(P)+DYP**2*EXP(P) -1./(Y+1.)**2
3      F = UXX + (1.+X*X)*UYY - Y*UX
3      RETURN
3      END
```

```
MACRO 10
*          2020221002000
1          POISSON   $   CONSTANT COEFFICIENTS   $   TWO DIMENSIONS
1          UXX$ + UYY$ = F(X,Y)
2          X=0. , U=TRUE(0.,Y)   $   X=1. , U=TRUE(1.,Y)
2          Y=0. , U=TRUE(X,0.)   $   Y=1. , U=TRUE(X,1.)
3      FUNCTION TRUE (X,Y)
3      COMMON /PARS/ A,B
3      DATA A,B /^A,^B/
3      BX3 = (B*X)**3
3      TRUE = SIN(X-Y+.5) + EXP(-Y*Y-(A*BX3/(1.+BX3))**2)
3      RETURN
3      END
3      FUNCTION F (X,Y)
3      COMMON /PARS/ A,B
3      BX  = (B*X)**3
3      DDH = 6.*X*B**3
3      DH = DDH*X/2.
3      H = 1. + DH*X/3.
3      G = A*(H-1.)
3      DG = A*DH
3      FE = G/H
3      DDG = A*DDH
3      DFE = DG/H - G*DH/(H**2)
3      DDFE = (H*DDG - 2.*DG*DH - G*DDH + 2.*G*DH*DH/H)/H**2
3      S = FE*FE
```

```
3       DS = 2.*FE*DFE
3       DDS = 2.*(FE*DDFE + DFE*DFE)
3       SINXY = SIN(X-Y+.5)
3       EE = EXP(-Y*Y-S)
3       UXX = -SINXY - EE*(DDS-DS*DS)
3       UYY = -SINXY - EE*2.*(1.-2.*Y*Y)
3       F = UXX + UYY
3       RETURN
3       END
-----------------------------------------
MACRO 11
*           2000201002000
1           TWO DIMENSIONS
1           UXX$ + UYY$ - EXP(TRUE(X,Y)) U$ = F(X,Y)
2           DIRICHLET
2           X=0. , U=TRUE(0.,Y)    $    X=1. , U=TRUE(1.,Y)
2           Y=0. , U=TRUE(X,0.)    $    Y=1. , U=TRUE(X,1.)
3       FUNCTION TRUE(X,Y)
3       TRUE = 10.*PHI(X)*PHI(Y) + ^A
3       RETURN
3       END
3       FUNCTION F(X,Y)
3       U = 10.*PHI(X)*PHI(Y) + ^A
3       W = EXP(U)
3       UXX = 10.*DDPHI(X)*PHI(Y)
3       UYY = 10.*PHI(X)*DDPHI(Y)
3       F = UXX + UYY - W*U
3       RETURN
3       END
3       FUNCTION PHI(Z)
3       PHI = (Z-1.)*Z*EXP(-100.*(Z-.5)**2)
3       RETURN
3       END
3       FUNCTION DDPHI(Z)
3       DH = -200.*(Z-.5)
3       DG = 2.*Z-1.
3       H = (Z-.5)*DH/2.
3       G = Z*(Z-1.)
3       DDPHI = (-200.*G + 2.*DH*DG + DH*DH*G + 2.)*EXP(H)
3       RETURN
3       END
-----------------------------------------
MACRO 12
*           2020221002000
1           POISSON    $    CONSTANT COEFFICIENTS    $    TWO DIMENSIONS
1           UXX$ + UYY$ = (.5*^A)*(.5*^A-1.)*(X*Y)**(.5*^A-2.)
1.                        *(X*X + Y*Y)
2           DIRICHLET
2           X=0. , U=TRUE(0.,Y)    $    X=1. , U=TRUE(1.,Y)
2           Y=0. , U=TRUE(X,0.)    $    Y=1. , U=TRUE(X,1.)
3       FUNCTION TRUE(X,Y)
3       TRUE = (X*Y)**(.5*^A)
3       RETURN
3       END
-----------------------------------------
MACRO 13
*           2000200002000
1           TWO DIMENSIONS
1           COEF1(X,Y) UXX$ + COEF2(X,Y) UYY$ + COEF3(X,Y) U
1.                     = F(X,Y)
2           DIRICHLET
2           X=0. , U=TRUE(0.,Y)    $    X=1. , U=TRUE(1.,Y)
2           Y=0. , U=TRUE(X,0.)    $    Y=1. , U=TRUE(X,1.)
3       FUNCTION COEF1(X,Y)
3       COMMON / PARS / A,B,C
3       DATA A,B,C /^A,^B,^C/
3       COEF1 = 2. + (Y-1.)*EXP(-A*Y**4)
3       RETURN
3       END
3       FUNCTION COEF2(X,Y)
3       COMMON / PARS / A,B,C
3       COEF2 = 1. + 1./(1.+(2.*X)**B)
```

```
3      RETURN
3      END
3      FUNCTION COEF3(X,Y)
3      COMMON / PARS / A,B,C
3      COEF3 = C*( X*(X-1.) + (Y-.3)*(Y-.7) )
3      RETURN
3      END
3      FUNCTION F(X,Y)
3      COMMON / PARS / A,B,C
3      F = 0.0
3      IF ((X .EQ. 0.0) .AND. (B .LE. 3.0))  RETURN
3      F1 = X + Y**2
3      P1 = (2.*X)**(B-1.)
3      DP1 = 2.*(B-1.)*(2.*X)**(B-2.)
3      DDP1 = 4.*(B-1.)*(B-2.)*(2.*X)**(B-3.)
3      DEN1 = 1. + P1
3      U1 = F1/DEN1
3      UXX1 = -(2.*(1. - F1*DP1/DEN1)*DP1 + F1*DDP1)/DEN1**2
3      UYY1 = 2./DEN1
3      E2 = EXP(-A*Y**4)
3      U2 = (1.+X)*(Y-1.)*E2
3      UXX2 = 0.0
3      UYY2 = (-4.*A*E2*((5.*Y-3.) - 4*A*Y**4*(Y-1.))*Y**2)*(1.+X)
3      C3 = COS(X*Y)
3      S3 = SIN(X*Y)
3      XY3 = X+Y
3      U3 = C*XY3*C3
3      UXX3 = -Y*(2.*C*S3 + Y*U3)
3      UYY3 = -X*(2.*C*S3 + X*U3)
3      COE1 = 2. + (Y-1.)*E2
3      COE2 = 1. + 1./(1.+2.*X*P1)
3      COE3 = C*(X*(X-1.)+(Y-.3)*(Y-.7))
3      F = COE1*(UXX1+UXX2+UXX3) + COE2*(UYY1+UYY2+UYY3) +COE3*(U1+U2+U3)
3      RETURN
3      END
3      FUNCTION TRUE(X,Y)
3      COMMON / PARS / A,B,C
3      TRUE = (X+Y*Y)/(1.+(2.*X)**(B-1.))
3     A      + (Y-1.)*(1.+X)*EXP(-A*Y**4)
3     B      + C*(X+Y)*COS(X*Y)
3      RETURN
3      END
-------------------------------------
MACRO 14
*          2000200002000
1          TWO DIMENSIONS
1          UXX $ + UYY $ + ^A/(Y+^E)UY = F(X,Y)
2          DIRICHLET
2          X =0. , U=TRUE(0.,Y)
2          X =1. , U=TRUE(1.,Y)
2          Y =0. , U=TRUE(X,0.)
2          Y =1. , U=TRUE(X,1.)
3      FUNCTION TRUE(X,Y)
3      TRUE = (Y**(^B) + COS(X*Y*Y)-1.)*X*X*(X-1.)**2
3      RETURN
3      END
3      FUNCTION F(X,Y)
3      F1 = Y**^B + COS(X*Y*Y) - 1.
3      DXF1 = -Y*Y*SIN(X*Y*Y)
3      DXXF1 = -Y**4*COS(X*Y*Y)
3      F2 = X*X*(X-1.)*(X-1.)
3      DXF2 = 4.*X**3 - 6.*X*X + 2.*X
3      DXXF2 = 12.*X*X - 12.*X + 2.
3      UX = DXF1*F2 + F1*DXF2
3      UXX = DXXF1*F2 + DXF1*DXF2 + DXF1*DXF2 + F1*DXXF2
3      DYF1 = ^B*Y**(^B-1.)-2.*Y*SIN(X*Y*Y)
3      DYYF1 = ^B*(^B-1.)*Y**(^B-2.)-2.*SIN(X*Y*Y)-
3     $         (2.*Y)**2*COS(X*Y*Y)
3      UY = DYF1*F2
3      UYY = DYYF1*F2
3      F = UXX + UYY + ^A/(Y+^E)*UY
3      RETURN
```

```
3     END
-----------------------------------
MACRO 15
*          2000221002002
1          CONSTANT COEFFICIENTS
1          UXX $ + 2.UYY $ + 3.UX$ -4.UY$ -U = F(X,Y)
2          DIRICHLET
2          X = 0. , U = TRUE(0.,Y)
2          X = 1. , U = TRUE(1.,Y)
2          Y = 0. , U = 0.
2          Y = 1. , U = 1. - .6*^A + ^A*ABS(X-.8)
3     FUNCTION TRUE(X,Y)
3     TRUE = Y*(1.-.8*(^A)**(2.-Y)+^A*ABS(X-.8)**(2.-Y)+X*Y*EXP(-X*Y)
3    $       *(Y-1.)
3     RETURN
3     END
3     FUNCTION F(X,Y)
3     Q = (2.-Y)*ALOG(ABS(X-.8))
3     P = 1.-.8*^A**(2.-Y) + ^A*EXP(Q)
3     DXQ = (2.-Y)*G(X)/ABS(X-.8)
3     DXXQ = -(2.-Y)*G(X)**2/(X-.8)**2
3     DYQ = -ALOG(ABS(X-.8))
3     DYYQ = 0.
3     DXP = ^A*DXQ*EXP(Q)
3     DXXP = ^A*DXXQ*EXP(Q)+^A*DXQ**2*EXP(Q)
3     DYP = .8*ALOG(^A)*EXP((2.-Y)*ALOG(^A))
3     DYYP = -.8*ALOG(^A)**2*EXP((2.-Y)*ALOG(^A))+
3          ^A*DYYQ*EXP(Q)+^A*DYQ**2*EXP(Q)
3     U = Y*P + X*EXP(-X*Y)*(Y*Y-Y)
3     UX = Y*DXP + (1.-X*Y)*(Y*Y-Y)*EXP(-X*Y)
3     UXX = Y*DXXP-Y*(Y*Y-Y)*EXP(-X*Y)*(2.-X*Y)
3     UY = P + EXP(-X*Y)*(-X*Y**3+X*Y**2+2.*X*Y-1.)
3     UYY = DYP -X*EXP(-X*Y)*(-X*Y**3+X*Y*Y+2.*X*Y-1.)
3    $     + EXP(-X*Y)*(-3.*X*Y*Y+2.*X*Y+2.*X)
3     F = UXX + 2.*UYY + 3.*UX -4.*UY - U
3     RETURN
3     END
3     FUNCTION G(X)
3     G = 1.
3     IF( X .LE. .8 ) G = -1.
3     RETURN
3     END
-------------------------------------------------------------
MACRO 16
*          2000221002002
1          TWO DIMENSIONS
1          UXX $ + (1.+X*Y)UYY $ COS(X)UX $ -EXP(-X)UY$+3.U=G(X,Y)
2          DIRICHLET
2          X = 0. , U = TRUE(X,Y)
2          X = 1. , U = TRUE(1.,Y)
2          Y = 0. , U = TRUE(X,0.)
2          Y = 1. , U = TRUE(X,1.)
3     FUNCTION TRUE(X,Y)
3     COMMON /PARS/ A,B
3     DATA A,B /^A,^B/
3     BX3 = (B*X)**3
3     TRUE = EXP(-Y*Y-(A*BX3/(1.+BX3))**2)+SIN(X-Y+.5)
3     RETURN
3     END
3     FUNCTION G(X,Y)
3     COMMON /PARS/ A,B
3     BX  = (B*X)**3
3     DDH = 6.*X*B**3
3     DH = DDH*X/2.
3     H = 1. + DH*X/3.
3     G = A*(H-1.)
3     DG = A*DH
3     DDG = A*DDH
3     FE = G/H
3     DFE = DG/H - G*DH/(H**2)
3     DDFE = (H*DDG - 2.*DG*DH - G*DDH + 2.*G*DH*DH/H)/H**2
3     S = FE*FE
```

```
3      DS = 2.*FE*DFE
3      DDS = 2.*(FE*DDFE + DFE*DFE)
3      SINXY = SIN(X-Y+.5)
3      EE = EXP(-Y*Y-S)
3      UXX = -SINXY - EE*(DDS-DS*DS)
3      UYY = -SINXY - EE*2.*(1.-2.*Y*Y)
3      UX = -2.*DFE*FE*EE + COS(X-Y+.5)
3      UY = -2.*Y*EE - COS(X-Y+.5)
3      F = UXX + (1.+X*Y)*UYY + COS(X)*UX-EXP(-X)*UY+3.*U
3      RETURN
3      END
-------------------------------------------------------
MACRO 17
*           2000200002000
1           SELF ADJOINT $ TWO DIMENSIONS
1           P^K(X,Y)UXX$+P^K(X,Y)UYY+DXP^K(X,Y)UX$+DYP^K(X,Y)UY=F(X,Y)
2           MIXED
2           X=0.,MIXED= UX =0.
2           X=1., MIXED= UX = 0.
2           Y=0., MIXED= U =TRUE(X,Y)
2           Y=1., MIXED= U =TRUE(X,Y)
3      FUNCTION TRUE(X,Y)
3      DATA PI/3.14
3      TRUE = COS(PI*X)*(P(Y)+1.)
3      RETURN
3      END
3      FUNCTION P1(X,Y)
3      UX = -PI*SIN(PI*X)*(P(Y)+1.)
3      UY = COS(PI*X)*D1P(Y)
3      H = (UX*UX + UY*UY)**.5
3      P1 = 1./(^A+^B*H)
3      RETURN
3      END
3      FUNCTION DXP1(X,Y)
3      COMMON /DATA/ PI
3      UX = -PI*SIN(PI*X)*(P(Y)+1.)
3      UY = COS(PI*X)*D1P(Y)
3      H = (UX*UX + UY*UY)**5
3      UXX = -PI*PI*COS(PI*X)*(P(Y)+1.)
3      UXY = -PI*SIN(PI*X)*D1P(Y)
3      HX = 1./H*(UX*UXX+UY*UXY)
3      DXP1 = -^B*HX/(^A+^B*H)**2
3      RETURN
3      END
3      FUNCTION DYP1(X,Y)
3      COMMON /DATA/ PI
3      UX = -PI*SIN(PI*X)*(P(Y)+1.)
3      UY = COS(PI*X)*D1P(Y)
3      H = (UX*UX + UY*UY)**5
3      UYY = COS(PI*X)*D2P(Y)
3      UXY = -PI*SIN(PI*X)*D1P(Y)
3      HX = 1./H*(UX*UXY+UY*UYY)
3      DYP1 = -^B*HY/(^A+^B*H)**2
3      RETURN
3      END
3      FUNCTION P2(X,Y)
3      COMMON /DATA/ PI
3      UX = -PI*SIN(PI*X)*(P(Y)+1.)
3      UY = COS(PI*X)*D1P(Y)
3      H = (UX*UX + UY*UY)**.5
3      P2 = EXP(H/(^A+^B*H))/H
3      RETURN
3      END
3      FUNCTION DXP2(X,Y)
3      COMMON /DATA/ PI
3      UX = -PI*SIN(PI*X)*(P(Y)+1.)
3      UY = COS(PI*X)*D1P(Y)
3      H = (UX*UX + UY*UY)**5
3      UXX = -PI*PI*COS(PI*X)*(P(Y)+1.)
3      UXY = -PI*SIN(PI*X)*D1P(Y)
3      HX = 1./H*(UX*UXX+UY*UXY)
3      DXP2 = -HX/H**2*EXP(H/(^A+^B*H))+1./H*(HX/(^A+^B*H))*
```

```
3     $       EXP(H/(^A+^B*H))
3      RETURN
3      END
3      FUNCTION DYP2(X,Y)
3      COMMON /DATA/ PI
3      UX = -PI*SIN(PI*X)*(P(Y)+1.)
3      UY = COS(PI*X)*D1P(Y)
3      H = (UX*UX + UY*UY)**.5
3      UYY = COS(PI*X)*D2P(Y)
3      UXY = -PI*SIN(PI*X)*D1P(Y)
3      HY = 1./H*(UX*UXY+UY*UYY)
3      DYP2 = -HY/H**2*EXP(H/(^A+^B*H))+1./H*(HY*P1(X,Y)+H*DXP1(X,Y))
3     $      *EXP(H*P1(X,Y))
3      RETURN
3      END
3      FUNCTION P3(X,Y)
3      COMMON /DATA/ PI
3      UX = -PI*SIN(PI*X)*(P(Y)+1.)
3      UY = COS(PI*X)*D1P(Y)
3      H = (UX*UX + UY*UY)**.5
3      P3 = ^A/H*TANH(^B*H)
3      RETURN
3      END
3      FUNCTION TANH(X)
3      EXPX = EXP(X)
3      REXPX = 1./EXPX
3      TANH = (EXPX - REXPX)/(EXPX + REXPX)
3      RETURN
3      END
3      FUNCTION DXP3(X,Y)
3      COMMON /DATA/ PI
3      UX = -PI*SIN(PI*X)*(P(Y)+1.)
3      UY = COS(PI*X)*D1P(Y)
3      H = (UX*UX + UY*UY)**.5
3      UXX = -PI*PI*COS(PI*X)*(P(Y)+1.)
3      UXY = -PI*SIN(PI*X)*D1P(Y)
3      HX = 1./H*(UX*UXX+UY*UXY)
3      TERM = EXP(2.*^B*H)
3      DXP3 = -^A*HX/H**2*(TEMP-1.)/(TEMP+1.)
3     $       + ^A/H*4.*^B*HX*TEMP/(TEMP+1.)**2
3      RETURN
3      END
3      FUNCTION DYP3(X,Y)
3      COMMON /DATA/ PI
3      UX = -PI*SIN(PI*X)*(P(Y)+1.)
3      UY = COS(PI*X)*D1P(Y)
3      H = (UX*UX + UY*UY)**.5
3      UYY = COS(PI*X)*D2P(Y)
3      UXY = -PI*SIN(PI*X)*D1P(Y)
3      HX = 1./H*(UX*UXY+UY*UYY)
3     $       EXP(H/(^A+^B*H))
3      TERM = EXP(2.*^B*H)
3      DYP3 = -^A*HY/H**2*(TEMP-1.)/(TEMP+1.)
3     $       + ^A/H*4.*^B*HY*TEMP/(TEMP+1.)**2
3      RETURN
3      END
3      FUNCTION F(X,Y)
3      COMMON /DATA/ PI
3      UX = -PI*SIN(PI*X)*(P(Y)+1.)
3      UXX = -PI*PI*COS(PI*X)*(P(Y)+1.)
3      UY  = COS(PI*X)*D1P(Y)
3      UYY = COS(PI*X)*D2P(Y)
3      F = P^K(X,Y)*(UXX+UYY)+DXP^K(X,Y)*UX+DYP^K(X,Y)UY
3      RETURN
3      END
3       FUNCTION P(X)
3       A = 1.
3       B = 0.
3       E = .15
3       X1 = .5 - E
3       X2 = .5 + E
3       IF(X .LT. X1) GO TO 1
```

```
3       IF(X .GT. X2) GO TO 2
3       DPHI = B - A
3       DX = X2 - X1
3       P = A + DPHI*(X-X1)**3/(DX**3)-3.*DPHI*(X-X1)**3*(X-X2)
3     $ /DX**4 + 6.*DPHI*(X-X1)**3*(X-X2)**2/DX**5
3       RETURN
31      P = A
3       RETURN
32      P = B
3       RETURN
3       END
3       FUNCTION D2P(X)
3       A = 1.
3       B = 0.
3       E = .15
3       X1 = .5 - E
3       X2 = .5 + E
3       IF(X .LT. X1) GO TO 1
3       IF(X .GT. X2) GO TO 1
3       DPHI = B - A
3       DX = X2 - X1
3       C3 = DPHI/DX**3
3       C4 = -3.*DPHI/DX**4
3       C5 = 6.*DPHI/DX**5
3       D2P = 6.*C3*(X-X1)+6.*C4*(X-X1)*(X-X2)+
3     $       6.*C4*(X-X1)**2+6.*C5*(X-X1)*(X-X2)**2+
3     $       12.*C5*(X-X1)**2*(X-X2)
3     $   +       2.*C5*(X-X1)**3
3       RETURN
31      D2P = 0.
3       RETURN
3       END
3       FUNCTION P(X)
3       A = 1.
3       B = 0.
3       E = .15
3       X1 = .5 - E
3       X2 = .5 + E
3       IF(X .LT. X1) GO TO 1
3       IF(X .GT. X2) GO TO 2
3       DPHI = B - A
3       DX = X2 - X1
3       D1P =  3.*DPHI*(X-X1)**2/(DX**3)-9.*DPHI*(X-X1)**2*(X-X2)
3     $ /DX**4 +18.*DPHI*(X-X1)**2*(X-X2)**2/DX**5
3     $-3.*DPHI*(X-X1)**3/DX**4
3     $+12.*DPHI*(X-X1)**3*(X-X2)/DX**5
3       RETURN
31      P = 0.
3       RETURN
32      P = 0.
3       RETURN
3       END
```

```
MACRO 18
*           2000221002002
1           CONSTANT COEFFICIENTS   $   TWO DIMENSIONS
1           UXX$ + UYY$ - ^A U  =  F(X,Y)
2           DIRICHLET
2           X=0.,U=TRUE(X,Y)
2           X=1.,U=TRUE(X,Y)
2           Y=0.,U=TRUE(X,Y)
2           Y=1.,U=TRUE(X,Y)
3       FUNCTION TRUE(X,Y)
3       TRUE = COS(^B*Y)+SIN(^B*(X-Y))
3       RETURN
3       END
3       FUNCTION F(X,Y)
3       BXMY = ^B*(X-Y)
3       B2 = ^B*^B
3       UXX = -B2*SIN(BXMY)
3       U = COS(^B*Y)+SIN(BXMY)
3       UYY = -B2*U
```

```
3       F = UXX + UYY - ^A*U
3       RETURN
3       END
```